

ON THE EXISTENCE OF W_p^2 SOLUTIONS FOR FULLY NONLINEAR ELLIPTIC EQUATIONS UNDER EITHER RELAXED OR NO CONVEXITY ASSUMPTIONS

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ABSTRACT. We establish the existence of solutions of fully nonlinear elliptic second-order equations like $H(v, Dv, D^2v, x) = 0$ in smooth domains without requiring H to be convex or concave with respect to the second-order derivatives. Apart from ellipticity nothing is required of H at points at which $|D^2v| \leq K$, where K is any given constant. For large $|D^2v|$ some kind of relaxed convexity assumption with respect to D^2v mixed with a VMO condition with respect to x are still imposed. The solutions are sought in Sobolev classes. We also establish the solvability without almost any conditions on H , apart from ellipticity, but of a “cut-off” version of the equation $H(v, Dv, D^2v, x) = 0$.

1. INTRODUCTION AND MAIN RESULTS

The first object, we deal with in this paper, is the equation

$$H[v](x) := H(v(x), Dv(x), D^2v(x), x) = 0 \quad (1.1)$$

considered in subdomains of \mathbb{R}^d , where $H(u, x)$ is a real-valued function defined for

$$u = (u', u''), \quad u' = (u'_0, u'_1, \dots, u'_d), \quad u'' \in \mathbb{S}, \quad x \in \mathbb{R}^d,$$

where \mathbb{S} is the set of symmetric $d \times d$ matrices. Let Ω be an open bounded subset of \mathbb{R}^d , fix $p > d$ and functions $\bar{G}, f \in L_p(\Omega)$, $\bar{G} \geq 0$. One of our main results implies that, for $d = 3$ and $\Omega \in C^2$, the equation $(a \wedge b = \min(a, b))$

$$H(D^2u, x) := \bar{G}(x) \wedge |D_{12}u| + \bar{G}(x) \wedge |D_{23}u| + \bar{G}(x) \wedge |D_{31}u| + 3\Delta u - f(x) = 0 \quad (1.2)$$

in Ω with zero boundary condition has a unique solution $u \in W_p^2(\Omega)$. Recall that $W_p^2(\Omega)$ denotes the set of functions v defined in Ω such that v , Dv , and D^2v are in $L_p(\Omega)$. Observe that H in (1.2) is neither convex nor concave with respect to D^2u . So far, there are only two approaches to such equations: the theory of (L_p) viscosity solutions and the theory of stochastic differential

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games, provided H has a somewhat special form. The past experience shows that it is hard to expect getting sharp quantitative results using probability theory. On the other hand, the theory of viscosity solutions indeed produced some remarkable quantitative results. However, to the best of the author's knowledge the result stated above about (1.2) is either very hard to obtain by using the theory of (L_p) viscosity solutions or is just beyond it, at least at the current stage. It seems that the best information, that theory provides at the moment, is the existence of the maximal and minimal L_p -viscosity solution (see [7]), no uniqueness of L_p -viscosity solutions can be inferred for (1.2) and no regularity apart from the classical C^α -regularity (see [3]).

Fix some constants $K_0, K_F \in [0, \infty)$, $\delta \in (0, 1]$, and an increasing continuous function $\omega(r)$, $r \geq 0$, such that $\omega(0) = 0$.

Assumption 1.1. The function $H(u, x)$ is measurable.

Next, we assume that there are two functions $F(u, x) = F(u'', x)$ and $G(u, x)$ such that

$$H = F + G.$$

Example 1.1. One can take $F(u'', x) = H(0, u'', x)$ and $G = H - F$. Since we will require later that $F(0, x) = 0$, one can then take $F(u'', x) = H(0, u'', x) - H(0, x)$ and $G = H - F$. However, we are not bound by these options.

The following assumptions contain parameters $\hat{\theta}, \theta \in (0, 1]$ which are specified later in our results.

Assumption 1.2. For $u'' \in \mathbb{S}$, $u' \in \mathbb{R}^{d+1}$, and $x \in \mathbb{R}^d$ we have

$$|G(u', u'', x)| \leq \hat{\theta}|u''| + K_0|u'| + \bar{G}(x).$$

Set

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \quad B_r = B_r(0),$$

and for Borel $\Gamma \subset \mathbb{R}^d$ denote by $|\Gamma|$ the volume of Γ . Also set

$$\mathbb{S}_\delta = \{a \in \mathbb{S} : \delta|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d\}.$$

Recall that Lipschitz continuous functions are almost everywhere differentiable.

Assumption 1.3. (i) The function F is Lipschitz continuous with respect to u'' with Lipschitz constant K_F , measurable with respect to x , and

$$F(0, x) \equiv 0.$$

Moreover, there exist $R_0 \in (0, 1]$ and $t_0 \in [0, \infty)$ such that for any $r \in (0, R_0]$ and $z \in \Omega$ one can find a *convex* function $\bar{F}(u'') = \bar{F}_{z,r}(u'')$ (independent of x) such that

(ii) We have $\bar{F}(0) = 0$ and at all points of differentiability of \bar{F} we have $(\bar{F}_{u''_{ij}}) \in \mathbb{S}_\delta$;

(iii) For any $u'' \in \mathbb{S}$ with $|u''| = 1$, we have

$$\int_{B_r(z) \cap \Omega} \sup_{t > t_0} t^{-1} |F(tu'', x) - \bar{F}(tu'')| dx \leq \theta |B_r(z) \cap \Omega|. \quad (1.3)$$

Assumption 1.4. We have $\Omega \in C^2$, and we are given a function $g \in W_p^2(\Omega)$.

We only consider $p > d$ and then by embedding theorems the functions of class $W_p^2(\Omega)$ admit modifications that are continuous in $\bar{\Omega}$ along with their first derivatives. We will always have in mind such modifications.

Here are our first two main results. The first one is an a priori estimate. Observe that in Theorem 1.1 we do not even assume that equation (1.1) is elliptic.

Theorem 1.1. *Under the above assumptions there exists a constant $\theta \in (0, 1]$, depending only on d, p, δ , and Ω , and a constant $\hat{\theta} \in (0, 1]$, depending only on $K_0, K_F, d, p, \delta, R_0$, and Ω , such that, if Assumptions 1.3 and 1.2 are satisfied with these θ and $\hat{\theta}$, respectively, then for any $u \in W_d^2(\Omega)$, that satisfies (1.1) in Ω (a.e.) and equals g on $\partial\Omega$, we have*

$$\|u\|_{W_p^2(\Omega)} \leq N(\|\bar{G}\|_{L_p(\Omega)} + \|g\|_{W_p^2(\Omega)} + \|u\|_{L_p(\Omega)}) + Nt_0, \quad (1.4)$$

where N depends only on $K_0, K_F, d, p, \delta, R_0$, and Ω .

In Section 8 we will see that Theorem 1.1 is an easy generalization of the estimate in Theorem 2.1 of [11]. However, in [11] the ellipticity of equation (1.1) is assumed from the start. It is also supposed and used that Assumption 1.5 below is satisfied with $\omega(t) = K_0 t$, and thus N in the estimate corresponding to (1.4) in Theorem 2.1 of [11] depends on the constant of Lipschitz continuity of H with respect to u' . We will see that, actually, it is independent.

To have the solvability we need ellipticity and more regularity of H .

Assumption 1.5. (i) The function $H(u, x)$ is Lipschitz continuous with respect to u'' , and at all points of differentiability of H with respect to u'' we have $D_{u''}H \in \mathbb{S}_\delta$;

(ii) The function $H(u, x)$ is nonincreasing with respect to u'_0 and

$$|H(u', u'', x) - H(v', u'', x)| \leq \omega(|u' - v'|)$$

for all u, v , and x .

Theorem 1.2. *Suppose that the above assumptions are satisfied and, moreover, Assumptions 1.2 and 1.3 are satisfied with $\hat{\theta}$ and θ from Theorem 1.1.*

Then for any $g \in W_p^2(\Omega)$ there exists $u \in W_p^2(\Omega)$ satisfying (1.1) in Ω (a.e.) and such that $u = g$ on $\partial\Omega$.

Theorem 1.2 is a generalization of the existence part of Theorem 2.1 of [11]. Again the whole point is that here we do not assume that H is Lipschitz in u' , which makes some arguments more involved. In particular, in [11] the existence is first proved when $g = 0$ and then the unknown u is replaced

with $u - g$. In our present case this would cause the last N in (1.4) to also depend on $\|g\|_{W_p^2(\Omega)}$. Then, the results of [11] are based on Theorem 1.1 of [10] in which the assumption of the Lipschitz continuity of H with respect to u' was indispensable. We abandon this assumption and then we have to show that the corresponding counterpart of Theorem 1.1 of [10], which is our third main result, Theorem 1.4, still holds.

These results are natural continuation of the results in [11]. Some discussion and example of applications of such results can be found there. In particular, an application to Isaacs equations is given.

Remark 1.1. In case of equation (1.2), to prove what is claimed after it, it suffices to take $F(D^2u, x) = \bar{F}(D^2u) = 3\Delta u$, and then all our assumptions are obviously satisfied. In particular, Assumption 1.5 (i) is satisfied since $H_{u''_{ii}} = 3$, $i = 1, 2, 3$, $|H_{u''_{12}}| = |H_{u''_{21}}|$, $|H_{u''_{23}}| = |H_{u''_{32}}|$, $|H_{u''_{31}}| = |H_{u''_{13}}| \leq 1$.

In the literature, interior $W_p^{2,p} > d$, *a priori* estimates for a class of fully nonlinear uniformly elliptic equations in \mathbb{R}^d of the form (1.1) in terms of viscosity solutions were first obtained by Caffarelli in [1] (1989) (see also [2] (1995)). Adapting his technique, similar interior *a priori* estimates were proved by Wang [19] (1992) for parabolic equations. In the same paper, a boundary estimate is stated but without proof; see Theorem 5.8 there. By exploiting a weak reverse Hölder's inequality, the result of [1] was sharpened by Escauriaza in [6] (1993), who obtained the interior W_p^2 -estimate for the same equations allowing $p > d - \varepsilon$, with a small constant ε depending only on the ellipticity constant and d .

The above cited works are quite remarkable in one respect—they do not suppose that H is convex or concave in D^2u . But they only show that to prove *a priori* estimates it suffices to prove the interior C^2 -estimates for “harmonic” functions. However, up to now, these estimates are only known under convexity assumptions.

Generally, having *a priori* estimates does not guarantee that there are existence results. Say, if $d = 1$ and equation is

$$u'' - (2u'')I_{|u''| \leq 5} - u = 1 \quad \text{in } (-1, 1).$$

a priori W_p^2 -estimates are trivial since $|(2u'')I_{|u''| \leq 5}| \leq 10$. However, the solvability is quite questionable.

Also obtaining boundary W_p^2 estimate by using the theory of viscosity solutions turned out to be extremely challenging and only in 2009, twenty years after the work of Caffarelli, Winter [20] proved the solvability in $W_p^2(\Omega)$ of equation (1.1) with Dirichlet boundary condition in $\Omega \in C^{1,1}$.

It is also worth noting that a solvability theorem in the space $W_{p, \text{loc}}^{1,2}(Q) \cap C(\bar{Q})$ can be found in M. G. Crandall, M. Kocan, A. Święch [4] (2000) for the boundary-value problem for fully nonlinear parabolic equations. The above mentioned existence results of [4] and [20] are proved under the assumption that H is convex in D^2v and in all papers mentioned above a small oscillation

assumption in the integral sense is imposed on the operators; see below. The above cited works are performed in the framework of viscosity solutions.

Caffarelli and Cabré [2] consider equations

$$H(D^2u, x) = f(x)$$

with $H(0, x) = 0$. They, and a very many other authors after them, introduced

$$\beta(x, x_0) = \sup_{u'' \in \mathbb{S}} \frac{|H(u'', x) - H(u'', x_0)|}{|u''|}.$$

and require that for all $B_r(x_0) \subset B_1$

$$\int_{B_r(x_0)} \beta^d(x, x_0) dx \leq \theta^d(p), \quad (1.5)$$

where $p > d$ and $\theta(p)$ is small enough. Then they prove, under an additional assumption on $H(u'', 0)$, that the $W_p^2(B_{1/2})$ -norm of the solution can be controlled. It is easily shown that $\theta(p) \downarrow 0$ as $p \rightarrow \infty$ implies that $H(u'', x)$ is just a continuous function of x .

Condition (1.5) might look like our condition (1.3). Therefore, it is important to emphasize that they are quite different. For instance, in the case of equation (1.2), if $\bar{G} \leq 1$, condition (1.5) implies that

$$\left| \int_{B_r(x_0)} G(x) dx - G(x_0) \right| \leq \theta(p),$$

where the integral is a uniformly continuous function of x_0 . According to Remark 1.1 we do not need any conditions on \bar{G} apart from $\bar{G} \in L_p(\Omega)$.

Probably, even better illustration of the difference gives the example of *linear* equations when $H[u] = a^{ij} D_{ij} u$. Then condition (1.5) is equivalent to the fact that $a^{ij}(x)$ are uniformly sufficiently close to uniformly continuous ones, and our condition is satisfied if, say $a^{ij} \in VMO$.

Also the additional assumption in [1], [2] on $H(u'', x)$ alluded to above, in the case of equation (1.2), includes the requirement that, for $f = 0$, (1.2) with $\bar{G}(0)$ in place of $\bar{G}(x)$ admit a solution of class $C^2(B_{1/2})$ for any continuous boundary data. So far, we have no idea whether this happens indeed and this is a very challenging problem.

Observe that in [4] and [20] the assumption that H is convex in u'' is needed because for such equations with additional regularity assumption the solvability in $C^{2+\alpha}$ is known and one can approximate the solutions with more regular ones in a usual way.

Our approach to construct the approximations is different and is based on our third main result before which we introduce some notation and assumptions.

Here is Theorem 3.1 of [9].

Theorem 1.3. *There exists an integer $m = m(\delta, d) > d$, a set*

$$\Lambda = \{l_{\pm 1}, \dots, l_{\pm m}\} \subset \bar{B}_1 \subset \mathbb{R}^d, \quad (1.6)$$

and there exists a constant

$$\hat{\delta} = \hat{\delta}(\delta, d) \in (0, \delta/4]$$

such that the coordinates of l_k are rational numbers and

- (a) We have $l_{-k} = -l_k$, $k = 1, \dots, m$, $e_i, (1/2)(e_i + e_j) \in \Lambda$, $i \neq j$, $i, j = 1, \dots, d$, where e_1, \dots, e_d is the standard orthonormal basis of \mathbb{R}^d ;
(b) There exist real-analytic functions $\lambda_{\pm 1}(a), \dots, \lambda_{\pm m}(a)$ on $\mathbb{S}_{\delta/4}$ such that for any $a \in \mathbb{S}_{\delta/4}$

$$a = \sum_{|k|=1}^m \lambda_k(a) l_k l_k^*, \quad \hat{\delta}^{-1} \geq \lambda_k(a) \geq \hat{\delta}, \quad \forall k. \quad (1.7)$$

For $z'' = (z''_{\pm 1}, \dots, z''_{\pm m})$ introduce

$$\mathcal{P}(z'') = \max_{\substack{\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1} \\ |k|=1, \dots, m}} \sum_{|k|=1}^m a_k z''_k, \quad (1.8)$$

and for $u'' \in \mathbb{S}$ define

$$P(u'') = \mathcal{P}(\langle u'' l_{\pm 1}, l_{\pm 1} \rangle, \dots, \langle u'' l_{\pm m}, l_{\pm m} \rangle), \quad (1.9)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^d .

Remark 1.2. Observe (cf. Section 9) that $P(u'')$ is Lipschitz continuous and at all points of its differentiability, for any $\xi \in \mathbb{R}^d$, we have

$$\min_{\substack{\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1} \\ |k|=1, \dots, m}} \sum_{|k|=1}^m a_k \langle l_k, \xi \rangle^2 \leq \langle \xi, P_{u''}(u'') \xi \rangle \leq \max_{\substack{\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1} \\ |k|=1, \dots, m}} \sum_{|k|=1}^m a_k \langle l_k, \xi \rangle^2.$$

Since, $e_i \in \Lambda \subset \bar{B}_1$, we see that there is a $\bar{\delta} = \bar{\delta}(d, \delta) \in (0, \delta]$ such that $P_{u''}(u'') \in \mathbb{S}_{\bar{\delta}}$.

For smooth enough functions $u(x)$ introduce

$$P[u](x) = P(D^2 u(x)).$$

Assumption 1.6. We are given a domain $\Omega \subset \mathbb{R}^d$ which is bounded and satisfies the exterior ball condition. We are also given a function $g \in C^{1,1}(\mathbb{R}^d)$.

By the exterior ball condition we mean that for any $x_0 \in \partial\Omega$ there exists a closed ball of radius $\rho(\Omega)$ (independent of x_0 and ≤ 1) centered outside Ω and having x_0 as the only common point with $\bar{\Omega}$.

Assumption 1.7. (i) Assumptions 1.1 and 1.5 are satisfied.

(ii) The number

$$\bar{H} := \sup_{u', x} (|H(u', 0, x)| - K_0 |u'|) \quad (\geq 0)$$

is finite.

Fix a constant $K \geq 0$ and consider the equation

$$\max(H[v], P[v] - K) = 0 \quad \text{in } \Omega \quad (\text{a.e.}) \quad (1.10)$$

with boundary condition $v = g$ on $\partial\Omega$. Here is the major result concerning (1.10). It generalizes Theorem 1.1 of [10] by relaxing the requirement of the Lipschitz continuity of H with respect to u' to just continuity. As there, we do not assume any regularity of the dependence of H on x and still obtain solutions with locally bounded second-order derivatives. Set

$$\rho_\Omega(x) = \text{dist}(x, \mathbb{R}^d \setminus \Omega).$$

Theorem 1.4. *Let Assumptions 1.6 and 1.7 be satisfied. Then*

(i) *equation (1.10) with boundary condition $v = g$ on $\partial\Omega$ has a solution $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$ such that,*

$$|v| \leq N(\bar{H} + \|g\|_{C(\Omega)}), \quad (1.11)$$

in Ω , where N is a constant depending only on d, δ, K_0 , and $\text{diam}(\Omega)$, and, for all $i, j = 1, \dots, d$,

$$|D_i v|, \rho_\Omega |D_{ij} v| \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\mathbb{R}^d)}) \quad \text{in } \Omega \quad (\text{a.e.}), \quad (1.12)$$

where N is a constant depending only on Ω, K_0, Λ , and δ (in particular, N is independent of ω);

(ii) *for $p > d$ and any such solution*

$$\|v\|_{W_p^2(\Omega)} \leq N_p(\bar{H} + K + \|g\|_{C^{1,1}(\mathbb{R}^d)}), \quad (1.13)$$

where N_p is a constant depending only on p, Ω, K_0, Λ , and δ .

Results of this kind, valid for H with no concavity or convexity assumptions, have independent interest and have already been used in [16] to show that the value functions in stochastic differential games in Ω admit approximations of order $1/K$ by functions whose second-order derivatives by magnitude are of order K in any compact subdomain of Ω . They were also used to show that solutions of Isaacs equations are in $C^{1+\alpha}$ if the coefficients are in VMO (see [14]), and that in a rather general case solutions of elliptic and parabolic Isaacs equations admit unique viscosity solutions that can be approximated, by using finite-difference equations, with algebraic rate of convergence with respect to the mesh size (see [15] and [17]).

Here we use Theorem 1.4 to prove Theorem 1.2.

Remark 1.3. Generally, in the situations of Theorem 1.2 and 1.4 there is no uniqueness. For instance, in the one-dimensional case each of the equations $D^2u - \sqrt{12|Du|} = 0$, $\max(D^2u - \sqrt{12|Du|}, 2(D^2u)^+ - (1/2)(D^2u)^- - 7) = 0$ for $x \in (-1, 1)$ with zero boundary data has two solutions: one is identically equal to zero and the other one is $1 - |x|^3$.

To guarantee uniqueness, it suffices to assume that $\omega(t) = N_0 t$, where N_0 is a constant. This follows from the fact that if u, v are solutions of, say

(1.10) with the properties described in Theorem 1.4 (i), then (cf. Section 9) the function $w = u - v$ equals zero on $\partial\Omega$ and in Ω satisfies the equation

$$a^{ij}D_{ij}w + b^iD_iw - cw = 0,$$

where (a^{ij}) is an \mathbb{S}_δ -valued function, $|(b^i)| \leq N_0$, $0 \leq c \leq N_0$. Then the equality $w = 0$ follows from the Aleksandrov maximum principle.

As we have pointed out already, the above results are close to those in [11] and [10], where the assumptions are stronger than here. The major difference between the assumptions in [11] and [10] and in the present article is that we do not assume that H is Lipschitz continuous with respect to u' with Lipschitz constant independent of u'', x . Thus, we considerably enlarge the number of equations for which the existence of solutions is guaranteed.

Abandoning the Lipschitz continuity with respect to u' causes some serious complications in the proof of our main start-up result: Theorem 1.4. In [11] we used an auxiliary cut-off equation

$$\max(H[v], P(u, Du, D^2u) - K) = 0 \quad (1.14)$$

and used a result, proved in [10], that such equations have solutions with locally bounded second-order derivatives. The reason why it happens is that, owing to the construction of P , on the set where

$$H(v(x), Dv(x), D^2v(x), x) \geq P(u, Du, D^2u) - K, \quad (1.15)$$

we had $|v|, |Dv|, |D^2v| \leq N(K)$. On the complement of this set

$$P(u, Du, D^2u) - K = 0. \quad (1.16)$$

In addition, the constructed P in [10] (and here) is convex in v, Dv, D^2v , so that one knows how to deal with (1.16) in order to get an *a priori* estimate of D^2v on the set where (1.16) holds through the values of D^2v on the boundary of this set, where (1.15) holds. However, one cannot justify this line of arguments because one has to have a sufficiently smooth v from the very beginning.

Therefore, the argument in [10] uses finite-difference approximations of (1.14), but first one has to rewrite it so that only pure second-order derivatives with respect to a fixed (rather large but finite) family of vectors are involved. One needs this because after replacing second-order derivatives with second-order differences one wants to get a monotone finite-difference equation (that is, an equation for which the maximum principle is valid). We know how to do that, preserving ellipticity, monotonicity in u'_0 , and measurability or any kind of continuity with respect to x , only if H is boundedly inhomogeneous with respect to u, Du, D^2u . In the situation of [11] one can just replace H with a different one, which is boundedly inhomogeneous with respect to u, Du, D^2u , and *without* changing equation (1.14).

After obtaining a finite-difference scheme one, basically, repeats the above argument involving (1.15) and (1.16) dealing with the second-order finite differences instead of D^2v . After proving that the finite-difference versions

of (1.14) have solutions with the second-order finite differences bounded independently of the mesh size, we pass to the limit, as the mesh size goes to zero. Once a sufficiently good solution of (1.14) is secured, (a priori) $W_p^2(\Omega)$ -estimates for that solution become available owing to a different analytical line of arguments. The estimates turned out to be independent of K and sending $K \rightarrow \infty$ finishes the job.

In our present situation the auxiliary equation becomes

$$\max(H(v(x), Dv(x), D^2v(x), x), P(D^2v) - K) = 0, \quad (1.17)$$

because we can hope to rewrite H as a boundedly inhomogeneous function only with respect to u'' . This turns out to be possible but the measure of inhomogeneity becomes depending on u' , which seems not to allow to preserve the *monotonicity* in u'_0 of the corresponding finite-difference equations and also led to other complications in estimates. In particular, in our heuristic argument, we cannot conclude that $|D^2v|$ is bounded on the set where

$$H(v(x), Dv(x), D^2v(x), x) \geq P(D^2u) - K$$

since there is no control of $|Du|$, and we cannot get it by differentiating (1.17). Furthermore, while speaking about (1.14) above, we said that we estimate D^2v , where the opposite to (1.15) holds, through the values of D^2v on the boundary of this set, to simplify the matter, we did not mention that the boundary maybe part of $\partial\Omega$, where there is no hope to get estimates for D^2v or for second-order differences.

In contrast with a parabolic version of Theorem 1.4, presented in [12], where losing monotonicity with respect to u'_0 , generally should not cause much problems (we only get our constants depending on the time length of the cylinder in which the problem is considered), in the elliptic case it could just break the whole argument, and an additional way to circumvent this difficulty was needed.

The article is organized as follows. Sections 2 through 7 are devoted to the proof of Theorem 1.4. In Section 2 we present some existence results about the solvability of nonlinear elliptic *finite-difference* schemes written in terms of pure finite-differences. In Section 3 we cite a particular case of Theorem 5.1 of [10], yielding estimates for solutions of special nonlinear finite-difference schemes with *constant* coefficients. We apply them in Section 4 to obtain estimates for finite differences for solution of “cut-off” finite-difference equations (similar to (1.10)). Section 5 contains existence results and estimates for “cut-off” *differential* equations written in terms of pure second-order derivatives. In Section 6, which is central in the paper, we consider general equations of “cut-off” type, but with H which is of “restricted bounded inhomogeneity” (in terms of [9]) and is Lipschitz with respect to u' . In Section 7 we achieve the proof of Theorem 1.4 and in Section 8 we give the proof of Theorems 1.1 and 1.2. Finally, Section 9 is an appendix where, for the convenience of the reader, we present some well-known properties of Lipschitz continuous functions.

2. SOLVABILITY OF ELLIPTIC FINITE-DIFFERENCE EQUATIONS

Fix a constant $m \in \{1, 2, \dots\}$ and a bounded domain $\Omega \subset \mathbb{R}^d$, and for $h > 0$ define

$$\Omega^h = \{x : \rho_\Omega(x) > h\}, \quad \partial_h \Omega = \bar{\Omega} \setminus \Omega^h,$$

and let $\mu(\Omega)$ be the sup of h such that $\Omega^h \neq \emptyset$.

Assumption 2.1. We are given vectors $l_i \in \bar{B}_1 \subset \mathbb{R}^d$, $i = \pm 1, \dots, \pm m$, such that $l_{-i} = -l_i$ and $\text{Span } \Lambda = \mathbb{R}^d$, where $\Lambda = \{l_i, |i| = 1, \dots, m\}$.

For $h > 0$ and $l \in \mathbb{R}^d$ define $T_{h,l}\phi(x) = \phi(x + hl)$,

$$\delta_{h,l}\phi(x) = \frac{1}{h}[T_{h,l}\phi(x) - \phi(x)], \quad \Delta_{h,l}\phi(x) = \frac{1}{h^2}[\phi(x + hl) - 2\phi(x) + \phi(x - hl)].$$

Remark 2.1. The reason to use $-l_i$ along with l_i comes from the fact that in our calculations $-l_i$ would appear anyway.

Assumption 2.2. (i) We are given a function $H(z, x)$, $x \in \bar{\Omega}$,

$$z = (z', z''), \quad z' = (z'_0, z'_{\pm 1}, \dots, z'_{\pm m}) \in \mathbb{R}^{2m+1}, \quad z'' = (z''_{\pm 1}, \dots, z''_{\pm m}) \in \mathbb{R}^{2m},$$

which is Lipschitz continuous with respect to z for any x . At all point of its differentiability with respect to z introduce

$$a_j = a_j(z, x) = D_{z'_j} H(z, x), \quad b_j = b_j(z, x) = D_{z''_j} H(z, x), \quad j = \pm 1, \dots, \pm m,$$

$$c = c(z, x) = -D_{z'_0} H(z, x),$$

and at points of non-differentiability set $a_j = \delta$, $b_j = 0$, $c = 1$.

(ii) The above introduced functions and $H(0, x)$ are bounded, $|b_j| \leq N'$, $|j| = 1, \dots, m$, where N' is a constant, and

$$\delta^{-1} \geq a_j \geq \delta \tag{2.1}$$

for all indices and values of the arguments.

(iii) We have $c \geq 0$.

For any function v on \mathbb{R}^d and $h > 0$ define

$$\begin{aligned} H_h[v](x) &= H(v, \delta_h v, \delta_h^2 v, x) \\ &= H(v, \delta_{h,l_{\pm 1}} v, \dots, \delta_{h,l_{\pm m}} v, \Delta_{h,l_{\pm 1}} v, \dots, \Delta_{h,l_{\pm m}} v, x), \end{aligned} \tag{2.2}$$

where

$$\delta_h v = (\delta_{h,l_{\pm 1}} v, \dots, \delta_{h,l_{\pm m}} v), \quad \delta_h^2 v = (\Delta_{h,l_{\pm 1}} v, \dots, \Delta_{h,l_{\pm m}} v). \tag{2.3}$$

Fix a bounded function g on $\bar{\Omega}$ and consider the equation

$$H_h[v] = 0 \quad \text{in } \Omega^h \tag{2.4}$$

with boundary condition

$$v = g \quad \text{on } \partial_h \Omega. \tag{2.5}$$

Of course, we will only consider h such that $\Omega^h \neq \emptyset$ ($h \in (0, \mu(\Omega))$). The following simple result can be found in [18] or in [9] or else in [13] if $c \geq 1$. The proofs there are based on the method of successive iteration, and the boundedness of $H(0, x)$ is used in order to start this procedure.

Theorem 2.1. *Let the above assumptions be satisfied. Then for all sufficiently small $h > 0$ the problem (2.4)-(2.5) has a unique bounded solution v_h .*

To prove Theorem 2.1 in the general case we need the following, where

$$D_{l_j} = l_j^i D_i.$$

Lemma 2.2. *There exists a function $\Psi_0 \in C^\infty(\mathbb{R}^d)$ such that, in Ω , $\Psi_0 \geq 1$ and*

$$a_j D_{l_j}^2 \Psi_0 + \delta^{-1} \sum_j |D_{l_j} \Psi_0| \leq -2, \quad (2.6)$$

and, for all sufficiently small $h > 0$, in Ω^h

$$a_j \Delta_{h,l_j} \Psi_0 + \delta^{-1} \sum_j |\delta_{h,l_j} \Psi_0| \leq -1 \quad (2.7)$$

whenever $\delta^{-1} \geq a_j \geq \delta$.

The proof of this lemma is an elementary exercise. Indeed, one takes

$$\Psi_0(x) = 1 + \cosh(\mu R) - \cosh(\mu|x|),$$

where R is twice the radius of the smallest ball centered at the origin containing Ω and by straightforward computations using that $\text{Span}\{l_j\} = \mathbb{R}^d$ and $a_j > \delta$ one gets that (cf. the proof of Lemma 4.3)

$$a_j D_{l_j}^2 \Psi_0 + \delta^{-1} \sum_j |D_{l_j} \Psi_0| \leq -\mu^2 \cosh(\mu|x|) [\varepsilon - \mu^{-1} \delta^{-1} 2m] \leq -2,$$

in Ω if μ is large enough, where $\varepsilon = \varepsilon(\Lambda) > 0$. This yields (2.6).

Then one observes that (2.6) is almost preserved if we replace D_{l_j} and $D_{l_j}^2$ with Δ_{h,l_j} and δ_{h,l_j} , respectively, and choose $h > 0$ sufficiently small. This is true owing to the continuity of the derivatives of Ψ_0 in \mathbb{R}^d .

Proof of Theorem 2.1. Take Ψ_0 from Lemma 2.2, in which we replace δ with a smaller one δ_1 in order to have $|b_j| \leq \delta_1^{-1}$, and for $h > 0$ set

$$\bar{H}^h(z, x) = H(Z'_0(z, x), Z'_{\pm 1}(z, x), \dots, Z'_{\pm m}(z, x), Z''_{\pm 1}(z, x), \dots, Z''_{\pm m}(z, x), x),$$

where, $Z'_0(z, x) = z'_0 \Psi_0(x)$ and, for $|j| = 1, \dots, m$,

$$Z'_j(z, x) = z'_j T_{h,l_j} \Psi_0(x) + z'_0 \delta_{h,l_j} \Psi_0(x),$$

$$Z''_j(z, x) = z''_j \Psi_0(x) + z'_j \delta_{h,l_j} \Psi_0(x) + z'_{-j} \delta_{h,l_{-j}} \Psi_0(x) + z'_0 \Delta_{h,l_j} \Psi_0(x)$$

(no summation in j). Note the fundamental property of \bar{H}^h which follows by simple arithmetics: for any function u ,

$$\bar{H}_h^h[u] = H_h[u \Psi_0]. \quad (2.8)$$

In addition, if we denote by $(\bar{a}_j^h, \bar{b}_j^h, \bar{c}^h)(z, x)$ the functions corresponding to \bar{H}^h as in Assumption 2.2, then obviously the boundedness and nondegeneracy conditions in Assumption 2.2 will be satisfied, and, due to Lemma

2.2,

$$\bar{c}^h = c\Psi_0 - b_j\delta_{h,l_j}\Psi_0 - a_j\Delta_{h,l_j}\Psi_0 \geq 1,$$

where we dropped obvious values of the arguments for simplicity.

By what was said before the theorem, for sufficiently small $h > 0$, the equation $\bar{H}_h^h[u] = 0$ in Ω^h with boundary condition $u = g/\Psi_0$ has a unique bounded solution. It only remains to set $v = u\Psi_0$ and use (2.8). The theorem is proved.

Lemma 2.3 (comparison principle). *If $h > 0$ is sufficiently small, then for any bounded functions u, v on $\bar{\Omega}$ such that $u \leq v$ on $\partial_h\Omega$ and $H_h[u] \geq H_h[v]$ in Ω^h we have $u \leq v$ in $\bar{\Omega}$.*

If $c(z, x) \geq \varepsilon > 0$, the lemma is a particular case of Theorem 2.2 of [13]. In the general case it suffices to use the argument in the end of the proof of Theorem 2.1.

In the future we will need an estimate of v_h .

Theorem 2.4. *Fix a constant $K_0 \in [0, \infty)$ and in addition to Assumptions 2.1 and 2.2 suppose that the number*

$$\hat{H} = \sup_{x \in \Omega, z': z'_0=0} (|H(z', 0, x)| - K_0|z'|)$$

is finite. Then there is a constant N , depending only on K_0, δ, Λ , and $\text{diam}(\Omega)$, such that for sufficiently small $h > 0$

$$\sup_{\bar{\Omega}} |v_h| \leq N(\hat{H} + \sup_{\bar{\Omega}} |g|).$$

Remark 2.2. Under Assumption 2.2, owing to the Lipschitz continuity of H with respect to z (uniform with respect to x in light of Assumption 2.2 (ii)), there always exists a K_0 such that $\bar{H} < \infty$. In the future, however, it will be important not to tighten up K_0 to the Lipschitz continuity.

The proof of Theorem 2.4 is based on the following.

Lemma 2.5. *Under the assumptions of Theorem 2.4 let ϕ be a function on Ω . Then for any $h \in (0, \mu(\Omega))$ (that is such that $\Omega^h \neq \emptyset$), there exist bounded functions a_j, b_j, c, f , $|j| = 1, \dots, m$, in Ω^h (generally different from the ones in Assumption 2.2) such that in Ω^h*

$$\begin{aligned} \delta \leq a_j \leq \delta^{-1}, \quad |b_j| \leq K_0, \quad c \geq 0, \quad |f| \leq \hat{H}, \\ H(\phi, \delta_h\phi, \delta_h^2\phi, x) = a_k\Delta_{h,l_k}\phi + b_k\delta_{h,l_k}\phi - c\phi + f. \end{aligned} \quad (2.9)$$

Proof. Notice that, due to Assumption 2.2 (cf. Section 9),

$$\begin{aligned} H(\phi, \delta_h\phi, \delta_h^2\phi, x) &= H(\phi, \delta_h\phi, \delta_h^2\phi, x) - H(\phi, \delta_h\phi, 0, x) \\ &\quad + H(\phi, \delta_h\phi, 0, x) = a_k\Delta_{h,l_k}\phi + H(\phi, \delta_h\phi, 0, x), \end{aligned}$$

where a_k are some functions satisfying $\delta \leq a_k \leq \delta^{-1}$, and

$$H(\phi, \delta_h\phi, 0, x) = -c\phi + f$$

with bounded $c \geq 0$ and $f := H(0, \delta_h \phi, 0, x)$ satisfying

$$|f| \leq \hat{H} + K_0 \sum_{|k|=1}^m |\delta_{h,l_k} \phi|.$$

This property of f implies that there exist functions b_k , $|k| = 1, \dots, m$, with values in $[-K_0, K_0]$ and θ with values in $[-1, 1]$ such that

$$f(\delta_h \phi, x) = b_k \delta_{h,l_k} \phi + \theta \hat{H}.$$

Upon combining all the above we come to (2.9). The lemma is proved.

Proof of Theorem 2.4. In Lemma 2.2 reduce δ , if necessary, in such a way that the new one, say δ_1 , satisfy $K_0 \leq \delta_1^{-1}$, and then take Ψ_0 from that lemma and set

$$\phi = \Psi_0(\hat{H} + \sup_{\bar{\Omega}} |g|).$$

Then by using Lemmas 2.5 and 2.2 we see that for $h \in (0, h_0]$ on Ω^h

$$H(\phi, \delta_h \phi, \delta_h^2 \phi, x) = a_k \Delta_{h,l_k} \phi + b_k \delta_{h,l_k} \phi - c\phi + \theta \hat{H} \leq -(\bar{H} + \sup_{\bar{\Omega}} |g|) + \theta \hat{H} \leq 0.$$

Furthermore, $v^h = g \leq \phi$ on $\partial_h \Omega$. It follows by Lemma 2.3 that $v^h \leq \phi$ on $\bar{\Omega}$. By replacing ϕ with $-\phi$ we get that $v^h \geq -\phi$ on $\bar{\Omega}$. The theorem is proved.

3. SOME ESTIMATES FOR FINITE-DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

Take an $h \in (0, 1]$, let $m \geq 1$ be an integer and let $l_{\pm 1}, \dots, l_{\pm m}$ be some fixed vectors in \mathbb{R}^d such that $l_{-k} = -l_k$. Denote

$$\Lambda = \{l_k : k = \pm 1, \dots, \pm m\},$$

$$\Lambda_1 = \Lambda, \quad \Lambda_{n+1} = \Lambda_n + \Lambda, \quad n \geq 1, \quad \Lambda_\infty = \bigcup_n \Lambda_n, \quad \Lambda_\infty^h = h\Lambda_\infty.$$

Let Q^o be a nonempty finite subset of Λ_∞^h . Introduce

$$Q = Q^o \cup \{x + h\Lambda : x \in Q^o\}. \quad (3.1)$$

Let A be a closed bounded set of points

$$a = (a_{\pm 1}, \dots, a_{\pm m}) \in \mathbb{R}^{2m}.$$

Assumption 3.1. There is a constant $\delta \in (0, 1]$ such that for any $a \in A$ and all k we have $a_k = a_{-k}$ and $\delta \leq a_k \leq \delta^{-1}$.

Also let $f(a)$ be a real-valued continuous function defined on A and for

$$z'' = (z''_{\pm 1}, \dots, z''_{\pm m}) \in \mathbb{R}^{2m},$$

introduce

$$P(z'') = \max_{a \in A} \left(\sum_{|k|=1}^m a_k z''_k + f(a) \right).$$

For any function u on \mathbb{R}^d define

$$P_h[u](x) = P(\delta_h^2 u(x)), \quad (3.2)$$

where δ_h^2 is introduced in (2.3).

In connection with this notation a natural question arises as to why use l_k along with $l_{-k} = -l_k$ since $\Delta_{h,l_k} = \Delta_{h,l_{-k}}$ and

$$a_k \Delta_{h,l_k} = 2 \sum_{k \geq 1} a_k \Delta_{h,l_k}$$

owing to the assumption that $a_k = a_{-k}$. This is done for the sake of convenience of computations. For instance,

$$\Delta_{h,l_k}(uv) = u \Delta_{h,l_k} v + v \Delta_{h,l_k} u + (\delta_{h,l_k} u)(\delta_{h,l_k} v) + (\delta_{h,-l_k} u)(\delta_{h,-l_k} v)$$

(no summation in k). At the same time

$$a_k \Delta_{h,l_k}(uv) = u a_k \Delta_{h,l_k} v + v a_k \Delta_{h,l_k} u + 2 a_k (\delta_{h,l_k} u)(\delta_{h,l_k} v),$$

as if we were dealing with usual partial derivatives. Another, even more compelling, reason is mentioned in Remark 2.1.

Next, take a function $\eta \in C^\infty(\mathbb{R}^d)$ with bounded derivatives, such that $|\eta| \leq 1$ and set $\zeta = \eta^2$,

$$|\eta'(x)|_h = \sup_k |\delta_{h,l_k} \eta(x)|, \quad |\eta''(x)|_h = \sup_k |\Delta_{h,l_k} \eta(x)|,$$

$$\|\eta'\|_h = \sup_{\Lambda_\infty^h} |\eta'|_h, \quad \|\eta''\|_h = \sup_{\Lambda_\infty^h} |\eta''|_h.$$

Finally, let u be a function on \mathbb{R}^d which satisfies

$$P_h[u] = 0 \quad \text{in } Q^o \quad (3.3)$$

and

$$P_h[u] \leq 0 \quad \text{on } Q \setminus Q^o. \quad (3.4)$$

Here is a one-sided interior estimate of pure second-order differences of u .

Theorem 3.1. *There exists a constant $N = N(m, \delta) \geq 1$ such that for any $r = \pm 1, \dots, \pm m$ (recall that $a^\pm = (1/2)(|a| \pm a)$) in Q we have*

$$\zeta^2[(\Delta_{h,l_r} u)^-]^2 \leq \sup_{Q \setminus Q^o} \zeta^2[(\Delta_{h,l_r} u)^-]^2 + N(\|\eta''\|_h + \|\eta'\|_h^2) \bar{W}_r, \quad (3.5)$$

where

$$\bar{W}_r = \sup_Q (|\delta_{h,l_r} u|^2 + |\delta_{h,l_{-r}} u|^2). \quad (3.6)$$

This theorem is a particular case of Theorem 5.1 of [10].

4. ESTIMATES FOR FINITE-DIFFERENCE EQUATIONS OF CUT-OFF TYPE WITH VARIABLE COEFFICIENTS

Fix some constants $K_0, K \in [0, \infty)$. We use the notation and assumptions introduced in the beginning of Section 2, however, we append Assumptions 2.1 and 2.2 with the following.

Assumption 4.1. The coordinates of $l_k, |k| = 1, \dots, m$, are rational numbers.

Assumption 4.2. The number

$$\bar{H} = \sup_{x, z'} (|H(z', 0, x)| - K_0 |z'|)$$

is finite.

We also impose the following.

Assumption 4.3. The domain Ω is bounded and satisfies the exterior ball condition. We are given a function $g \in C^{1,1}(\mathbb{R}^d)$.

For $z'' \in \mathbb{R}^{2m}$ introduce

$$P(z'') = \max_{\substack{\delta/2 \leq a_k \leq 2\delta^{-1} \\ |k|=1, \dots, m}} \sum_{|k|=1}^m a_k z''_k \quad (4.1)$$

Observe that

$$H(z, x) \leq P(z'') - (\delta/2) \sum_{|k|=1}^m |z''_k| + K_0 |z'| + \bar{H}. \quad (4.2)$$

Indeed, it follows from Assumption 2.2 that (cf. Section 9)

$$H(z, x) - H(z', 0, x) = \sum_{|k|=1}^m a_k z''_k,$$

where $\delta \leq a_k \leq \delta^{-1}$ for all k . Hence,

$$H(z, x) - H(z', 0, x) + (\delta/2) \sum_{|k|=1}^m |z''_k| = \sum_{|k|=1}^m b_k z''_k,$$

where $b_k := a_k + (\delta/2) \text{sign } z''_k$ lie in $(\delta/2, \delta^{-1} + \delta/2) \subset (\delta/2, 2/\delta)$.

We need one more function

$$H_K(z, x) = \max(H(z, x), P(z'') - K).$$

Recall that $\delta_h^2 v$ and $\delta_h v$ are defined in (2.3) and for any function v on Ω set

$$H_{K,h}[v](x) = H_K(v(x), \delta_h v(x), \delta_h^2 v(x), x).$$

We will concentrate on sufficiently small h such that $\Omega^h \neq \emptyset$ and will consider the equation

$$H_{K,h}[v] = 0 \quad \text{in} \quad \Omega^h \quad (4.3)$$

with boundary condition

$$v = g \quad \text{on} \quad \partial_h \Omega. \quad (4.4)$$

By Theorem 2.1 for any sufficiently small $h > 0$ there exists a unique bounded solution $v = v_h$ of (4.3)–(4.4). By the way, we do not include K in the notation v_h since K is a fixed number. The solution of the PDE version of (4.3)–(4.4) will be obtained as the limit of a subsequence of v_h as $h \downarrow 0$. Therefore, we need to have appropriate bounds on v_h and the first- and second-order differences in x of v_h .

Observe that owing to Assumption 4.2, for $z'_0 = 0$,

$$\begin{aligned} -K_0|z'| - \bar{H} &\leq H(0, z'_{\pm 1}, \dots, z'_{\pm m}, 0, x) \leq H_K(0, z'_{\pm 1}, \dots, z'_{\pm m}, 0, x) \\ &= \max(H(0, z'_{\pm 1}, \dots, z'_{\pm m}, 0, x), -K) \leq K_0|z'| + \bar{H}, \end{aligned}$$

so that $\hat{H}_K \leq \bar{H}$, where \hat{H}_K is taken from Theorem 2.4 with H_K in place of H . Therefore, the following is a direct consequence of Theorem 2.4.

Lemma 4.1. *There is a constant N , depending only on Λ , K_0 , δ , and $\text{diam}(\Omega)$, such that for sufficiently small $h > 0$*

$$\sup_{\bar{\Omega}} |v_h| \leq N(\bar{H} + \|g\|_{C(\Omega)}). \quad (4.5)$$

Lemma 2.5 also yields the following useful result.

Lemma 4.2. *Let ϕ be a function on Ω . Then for any $h \in (0, \mu(\Omega))$ there exist bounded functions a_j, b_j, c, f , $|j| = 1, \dots, m$, on Ω^h such that on Ω^h*

$$\begin{aligned} \delta/2 \leq a_j \leq 2\delta^{-1}, \quad |b_j| \leq K_0, \quad c \geq 0, \quad |f| \leq \bar{H}, \\ H_{K,h}[\phi] = a_k \Delta_{h,l_k} \phi + b_k \delta_{h,l_k} \phi - c\phi + f. \end{aligned} \quad (4.6)$$

Below by N with occasional indices we denote various (finite) constants depending only on $\text{diam}(\Omega)$, $\rho(\Omega)$, $\mu(\Omega)$, $\Lambda = \{l_i\}$, K_0 , d , and δ , unless explicitly stated otherwise.

We need a barrier function. Recall that $\rho(\Omega) \leq 1$.

Lemma 4.3. *There exists a constant $\alpha > 0$, depending only on Λ , δ , $\rho(\Omega)$, and K_0 , such that for $\psi(x) = |x|^{-\alpha} - \rho^{-\alpha}(\Omega)$ we have*

$$a_k D_{l_k}^2 \psi(x) + b_k D_{l_k} \psi(x) - c\psi(x) \geq 1, \quad (4.7)$$

whenever $2\delta^{-1} \geq a_k \geq \delta/2$, $|b_k| \leq K_0$, $|k| = 1, \dots, m$, $|c| \leq K_0$, and $0 < |x| \leq 3$.

Proof. Observe that

$$a_k D_{l_k}^2 \psi = a^{ij} D_{ij} \psi,$$

where

$$a^{ij} = a_k l_k^i l_k^j.$$

For any $\lambda \in \mathbb{R}^d$ it holds that

$$a^{ij} \lambda_i \lambda_j = a_k (l_k, \lambda)^2 \geq \delta \sum_{k=1}^m (l_k, \lambda)^2 \geq \delta_1 |\lambda|^2,$$

$$a^{ij}\lambda_i\lambda_j \leq 2m\delta^{-1}|\lambda|^2 =: N_1|\lambda|^2,$$

where $\delta_1 > 0$ is a constant whose existence is guaranteed by the assumption that $\text{Span}\{l_j\} = \mathbb{R}^d$.

For α such that $(\alpha + 2)\delta_1 \geq N_1d + \alpha\delta_1/2$ we obtain that

$$a^{ij}D_{ij}\psi = \alpha|x|^{-\alpha-2}((\alpha + 2)|x|^{-2}a^{ij}x_ix_j - \text{tr}(a^{ij})) \geq (\delta_1/2)\alpha^2|x|^{-\alpha-2}.$$

One easily finishes the proof after noticing that

$$|b_k D_{l_k} \psi| = \alpha|x|^{-\alpha-2}|b_k(l_k, x)|, \quad |c\psi| \leq K_0(|x|^{-\alpha} + \rho^{-\alpha}(\Omega)).$$

The lemma is proved.

Lemma 4.4. *There is a constant N such that, for all sufficiently small $h > 0$,*

$$|v_h - g| \leq N(\bar{H} + \|g\|_{C^{1,1}(\Omega)})(\rho_\Omega \wedge 1) \quad (4.8)$$

on $\bar{\Omega}$.

Proof. Observe that (4.4) implies that (4.8) holds in $\partial_h\Omega$. Therefore, we may concentrate on proving (4.8) in Ω^h .

Take an $x_0 \in \Omega^h$ (so that $h < \rho_\Omega(x_0)$). If $\rho_\Omega(x_0) \geq 1$, then (4.8) holds at x_0 in light of Lemma 4.1. Hence we may assume that $\rho_\Omega(x_0) \leq 1$. Let y be a closest point to x_0 on $\partial\Omega$. Then the closures of the following two balls has only one common point y : one is the ball centered at x_0 with radius $\rho_\Omega(x_0)$ and the other is the ball of radius $\rho(\Omega)$, which lies outside Ω and is centered, say at y_0 . Then all three points x_0 , y , and y_0 lie on the same line. Therefore, without loss of generality, on the account of moving the origin, we may assume that $y_0 = 0$, the straight line passing through the origin and x_0 crosses $\partial\Omega$ at y , and $|y| = \rho(\Omega)$ (≤ 1 by assumption)

Then set

$$\phi = g - 2N_1(\bar{H} + \|g\|_{C^{1,1}(\Omega)})\psi,$$

where ψ is taken from Lemma 4.3 and the constant N_1 is such that in Ω^h

$$a_k\Delta_{h,l_k}g + b_k\delta_{h,l_k}g - cg + f \leq N_1(\bar{H} + \|g\|_{C^{1,1}(\Omega)})$$

as long as conditions (4.6) are satisfied.

We increase N_1 if necessary in order to have $v_h \leq \phi$ in $\Omega \setminus B_2$, the latter being possible owing to Lemma 4.1 and the fact that in $\Omega \setminus B_2$ we have

$$-\psi \geq \rho^{-\alpha}(\Omega) - 2^{-\alpha} \geq 1 - 2^{-\alpha}.$$

We also concentrate on $h > 0$ sufficiently small such that the finite-difference approximations of $D_{l_k}^2\psi$ and $D_{l_k}\psi$ are sufficiently close to these quantities in $B_3 \setminus B_{\rho(\Omega)}$, so that owing to Lemma 4.3

$$a_k\Delta_{h,l_k}\psi + b_k\delta_{h,l_k}\psi - c\psi \geq 1/2$$

in $B_3 \setminus B_{\rho(\Omega)}$.

Then for $\Omega_1 = \Omega \cap B_3$ (where $\psi < 0$), by using Lemma 4.2, we find that in Ω_1^h

$$\max(H(\phi, \delta_h\phi, \delta_h^2\phi, x), P_h[\phi] - K) \leq 0.$$

Also by the choice of N_1 , $v_h \leq \phi$ on $\partial_h \Omega_1$ (if $h < 1$). Hence by the comparison principle, $v_h \leq \phi$ in $\Omega_1 = \Omega \cap B_3$. By the choice of N_1 , this inequality, actually, holds in Ω , which yields the desired estimate of $v_h - g$ from above. Similarly one obtains it from below as well. The lemma is proved.

The proof of the lemma allows us to get control on the boundary behavior of v_h uniformly with respect to h in a relatively easy way. The way to treat finite-differences of v_h is much more involved.

For sufficiently small $h > 0$ (such that v_h is well defined) introduce

$$M_h(x) = \sum_{|k|=1}^m |\delta_{h,l_k} v_h(x)|, \quad \bar{M}_h = \sup_{\Omega^h} M_h,$$

$$G := \{x \in \Omega^h : (\delta/2) \sum_{k=1}^m |\Delta_{h,l_k} v_h(x)| > \bar{H} + K + K_0(|v_h(x)| + M_h(x))\}.$$
(4.9)

and observe a fundamental fact that thanks to (4.2) (cf. (1.16))

$$P_h[v_h] - K = 0 \quad \text{in } G. \quad (4.10)$$

Also observe that on $\Omega^h \setminus G$ we have

$$(\delta/2) \sum_{k=1}^m |\Delta_{h,l_k} v_h| \leq \bar{H} + K + K_0(|v_h| + M_h). \quad (4.11)$$

Lemma 4.5. *There is a constant N such that, for all sufficiently small $h > 0$ and $r = 1, \dots, m$, in Ω^h we have*

$$(\rho_\Omega - 2h) |\Delta_{h,l_r} v_h| \leq N(\bar{M}_h + \bar{H} + K + \|g\|_{C^{1,1}(\Omega)}). \quad (4.12)$$

Proof. Having in mind translations, we see that it suffices to prove (4.12) in $\Omega^{2h} \cap \Lambda_\infty^h$. Then fix r and define

$$Q^o := \Omega^{2h} \cap \Lambda_\infty^h \cap G.$$

In light of Assumption 4.1 (used for the first time), the set Q^o is finite, since the number of points in Λ_∞^h lying in any ball is finite, because for an appropriate integer I we have $I\Lambda_\infty^h \subset h\mathbb{Z}^d$.

For Q from (3.1), obviously, $Q \subset \Omega^h$. Next, if $x \in \Omega^{2h} \cap \Lambda_\infty^h$ is such that $x \notin Q^o$, then (4.11) is valid, in which case (4.12) holds.

Thus, we need only prove (4.12) on Q^o assuming, of course, that $Q^o \neq \emptyset$. We know that (4.10) holds and the left-hand side of (4.10) is nonpositive in $Q \setminus Q^o$ (as everywhere else in Ω^h).

To proceed further we use a simple fact that there exists a constant $N \in (0, \infty)$ depending only on d such that for any $\mu \in (0, \mu(\Omega)/2)$ there exists an $\eta_\mu \in C_0^\infty(\Omega)$ satisfying

$$\eta_\mu = 1 \quad \text{on } \Omega^{2\mu}, \quad \eta_\mu = 0 \quad \text{outside } \Omega^\mu,$$

$$|\eta_\mu| \leq 1, \quad |D\eta_\mu| \leq N/\mu, \quad |D^2\eta_\mu| \leq N/\mu^2.$$

By Theorem 3.1 in Q^o

$$[(\Delta_{h,l_r} v_h)^-]^2 \leq \sup_{Q \setminus Q^o} \eta_\mu [(\Delta_{h,l_r} v_h)^-]^2 + N\mu^{-2} \bar{M}_h^2.$$

While estimating the last supremum we will only concentrate on (sufficiently small h and)

$$\mu \in [2h, \mu(\Omega)/2],$$

when $\eta_\mu = 0$ outside Ω^{2h} . In that case, for any $y \in Q \setminus Q^o$, either $y \notin \Omega^{2h}$ implying that

$$\eta_\mu [(\Delta_{h,l_r} v_h)^-]^2(y) = 0,$$

or $y \in \Omega^{2h} \cap \Lambda_\infty^h$ but (4.11) holds at y .

It follows that, as long as $x \in Q^o$ and $\mu \in [2h, \mu(\Omega)/2]$, we have

$$\mu(\Delta_{h,l_r} v_h)^-(x) \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)} + \bar{M}_h). \quad (4.13)$$

Take

$$\mu = (\mu(\Omega)/2) \wedge \rho_\Omega(x),$$

which is bigger than $2h$ on Q^o . Also $\mu = \rho_\Omega(x)$ if $\rho_\Omega(x) \leq \mu(\Omega)/2$ and

$$\mu = \mu(\Omega)/2 \geq \rho_\Omega(x) \mu(\Omega)/(2 \operatorname{diam}(\Omega))$$

if $\rho_\Omega(x) \geq \mu(\Omega)/2$. This and (4.13) yield that on Q^o

$$\rho_\Omega(\Delta_{h,l_r} v_h)^-(x) \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)} + \bar{M}_h). \quad (4.14)$$

Obviously, one can replace ρ_Ω in (4.14) with $\rho_\Omega - 3h$ and as a result of all the above arguments we see that

$$(\rho_\Omega - 3h)(\Delta_{h,l_r} v_h)^- \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)} + \bar{M}_h) \quad (4.15)$$

holds in Q^o for any r whenever h is small enough.

Finally, since $P_h[v_h] \leq K$ in Ω^h , we have that

$$4\delta^{-1} \sum_{r=1}^m (\Delta_r v_h)^+ \leq \delta \sum_{r=1}^m (\Delta_r v_h)^- + K,$$

which after being multiplied by $(\rho_\Omega - 3h)^+$ along with (4.15) leads to (4.12) on Q^o . Thus, as is explained at the beginning of the proof, the lemma is proved.

Now we exclude \bar{M}_h from (4.12).

Lemma 4.6. *There is a constant N such that for all sufficiently small $h > 0$ the estimates*

$$|v_h|, |\delta_{h,l_k} v_h|, (\rho_\Omega - 2h)|\Delta_{h,l_k} v_h| \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)}) \quad (4.16)$$

hold in Ω^h for all k .

Proof. Owing to Lemmas 4.4 and 4.5, (4.16) would follow if we can prove that

$$|\delta_{h,l_k} v_h| \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\Omega)}) \quad (4.17)$$

in Ω^h for all k . Actually, estimate (4.17) is proved in Lemma 4.7 of [12] for the parabolic case by using interpolation inequalities, but the proof is valid word for word for the elliptic case as well. The lemma is proved.

Remark 4.1. Much of what is done above goes through for domains satisfying the exterior cone condition instead of the exterior ball condition. However, then it would be impossible to get the global discrete gradient estimate and to get estimates of $\Delta_{h,l_k} v_h$ in a closed form.

5. A PARTICULAR CASE OF ELLIPTIC EQUATIONS IN PURE DERIVATIVES

Fix some constants $m \in \{1, 2, \dots\}$ and $K_0, K, N' \in (0, \infty)$.

Assumption 5.1. We are given vectors $l_i \in \bar{B}_1 \subset \mathbb{R}^d$, $i = \pm 1, \dots, \pm m$, such that $l_{-i} = -l_i$, the coordinates of l_i 's are rational numbers, and

$$e_i, (1/2)(e_i \pm e_j) \in \Lambda := \{l_k : k = \pm 1, \dots, \pm m\}$$

for $i, j = 1, \dots, d$, $i \neq j$, where e_i 's form the standard orthonormal basis in \mathbb{R}^d .

Assumption 5.2. We are given a domain $\Omega \subset \mathbb{R}^d$ which is bounded and satisfies the exterior ball condition. We are also given a function $g \in C^{1,1}(\mathbb{R}^d)$.

The following assumption has some parts which are close to Assumption 2.2.

Assumption 5.3. (i) We are given a continuous function $H(z, x)$, $x \in \mathbb{R}^d$, $z = (z', z'')$, $z' = (z'_0, z'_{\pm 1}, \dots, z'_{\pm m}) \in \mathbb{R}^{2m+1}$, $z'' = (z''_{\pm 1}, \dots, z''_{\pm m}) \in \mathbb{R}^{2m}$, which is Lipschitz continuous with respect to $z \in \mathbb{R}^{4m+1}$ with Lipschitz constant N' for any $x \in \mathbb{R}^d$. At all point of its differentiability with respect to z introduce

$$a_j = a_j(z, x) = D_{z'_j} H, \quad c = c(z, x) = -D_{z'_0} H$$

and at points of non-differentiability set $a_j = \delta$ and $c = 1$.

(ii) The above introduced functions a_j and c satisfy

$$\delta^{-1} \geq a_j \geq \delta, \quad c \geq 0 \quad (5.1)$$

for all indices and values of the arguments.

(iii) The number

$$\bar{H} = \sup_{x, z'} (|H(z', 0, x)| - K_0 |z'|)$$

is finite.

(iv) The function $H(z, x)$ is Lipschitz continuous with respect to $x \in \mathbb{R}^d$ for any $z \in \mathbb{R}^{4m+1}$ and at any point x of its differentiability with respect to x

$$|D_x H(z, x)| \leq N'(1 + |z|). \quad (5.2)$$

(v) There is a constant $\rho_0 > 0$ and a function $H(z)$ such that

$$H(z, x) = H(z) \quad (5.3)$$

for all z if $\rho_\Omega(x) \leq \rho_0$ (recall that $\rho_\Omega = \rho_\Omega(x) = \text{dist}(x, \mathbb{R}^d \setminus \Omega)$).

For sufficiently smooth functions $u = u(x)$ introduce

$$\begin{aligned} H[u](x) &= H(u(x), D_{l_j} u(x), D_{l_k}^2 u(x), x) \\ &= H(u(x), D_{l_{\pm 1}} u(x), \dots, D_{l_{\pm m}} u(x), D_{l_{\pm 1}}^2 u(x), \dots, D_{l_{\pm m}}^2 u(x), x). \end{aligned}$$

Similarly we introduce $P[u]$, where P is taken from (4.1).

We will be interested in finding a solution $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$ of equation (1.10) in Ω with boundary data g . We will also be interested in obtaining the estimates

$$|v| \leq N(\bar{H} + \|g\|_{C(\partial\Omega)}) \quad \text{in } \Omega, \quad (5.4)$$

$$|v|, |D_i v|, \rho_\Omega |D_{ij} v| \leq N(\bar{H} + K + \|g\|_{C^{1,1}(\mathbb{R}^d)}) \quad \text{in } \Omega \quad (\text{a.e.}), \quad (5.5)$$

where $i, j = 1, \dots, d$, $D_{ij} v$ are generalized derivatives of v , and the constants \bar{H} and N are chosen appropriately.

Here is a pilot result, which will be gradually generalized to a very large extent in the subsequent sections. The first generalization is given by Theorem 5.3 in which the global Lipschitz condition of $H(z, x)$ with respect to z is replaced with the local one.

Theorem 5.1. *Under the above assumptions equation (1.10) with boundary condition $v = g$ on $\partial\Omega$ has a unique solution $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$. Furthermore, estimates (5.5) hold with N depending only on Ω , K_0 , Λ , and δ (in particular, N is independent of N' and ρ_0).*

Remark 5.1. The constant N in (5.5) is independent of N' and ρ_0 which enter Assumptions 5.3 (i), (iv), and (v). In Theorem 1.4 all these assumptions are dropped. Therefore, it is worth explaining how we use them here. Assumption 5.3 (i) will allow us to use the results about finite-difference equations while replacing pure second-order derivatives with second-order differences and Assumptions 5.3 (iv), (v) will allow us to show that the collection of solutions of finite-difference equations is almost equicontinuous in Ω in the sense specified in Lemma 5.2.

To prove Theorem 5.1 we need a lemma. But first of all we observe that, by standard arguments, if we have two solutions u and v , then in Ω (a.e.)

$$\begin{aligned} 0 &= \max(H[u], P[u] - K) - \max(H[v], P[v] - K) \\ &= a^{ij} D_{ij}(u - v) + b^i D_i(u - v) - c(u - v) \end{aligned}$$

(cf. Section 9), where $a = (a^{ij})$, $b = (b^i)$, and c are certain functions satisfying $a \in \mathbb{S}_{\delta/2}$, $|b| \leq N'$, $0 \leq c \leq N'$. By the Aleksandrov maximum principle $u - v = 0$ and hence uniqueness.

As is explained above, we are going to use finite differences in the proof of the existence and the estimates. Observe that Assumptions 2.1, 4.1, 4.2, and 4.3 are satisfied due to Assumptions 5.1, 5.2, and 5.3. Assumption 2.2 is also satisfied in light of Assumption 5.3, in particular, the fact that $H(0, x)$ is a bounded function follows from Assumption 5.3 (iii).

Thus, all the assumptions of Section 4 are satisfied and we can use the information obtained in this section about functions v_h defined in $\bar{\Omega}$ for sufficiently small $h > 0$ as unique bounded solutions of (4.3)-(4.4).

Estimates (4.16) are estimates of v_h on the translates of a multiple of \mathbb{Z}^d . They do not tell us anything about the difference $v_h(x) - v_h(y)$ if x and y cannot be connected by a broken line consisting of translates of hl_j , $|j| = 1, \dots, m$. However, this information is necessary if we want to let $h \downarrow 0$ and extract a subsequence of v_h converging everywhere in Ω . In the proof of the following result Assumptions 5.3 (iv) and (v) are crucial.

Lemma 5.2. *There exists a constant N such that for all sufficiently small $h > 0$ we have*

$$|v_h(x) - v_h(y)| \leq N(|x - y| + h) \quad (5.6)$$

whenever $x, y \in \bar{\Omega}$.

Proof. First we want to estimate $v_h(x) - v_h(y)$ when $|x - y| \leq h$. Take $x_0 \in \Omega$ and $h > 0$ such that $4h < \rho_\Omega(x_0)$. Denote by N_1 the right-hand side of (4.16) and introduce

$$D_H = \{(z, x) : |z'_k| \leq N_1, |z''_j| \leq N_1(\rho_\Omega(x_0) - 3h)^{-1}, \\ |k| = 0, \dots, m, |j| = 1, \dots, m\} \times \bar{B}_h(x_0).$$

We claim that, on the account of our assumptions, for all sufficiently small $h > 0$, there is a constant N_2 , independent of x_0 , such that, at all points of differentiability of H with respect to (z, x) belonging to the interior of D_H , we have

$$\delta/2 \leq D_{z''_j} H(z, x) \leq 2\delta^{-1}, \quad |j| = 1, \dots, m, \quad 0 \leq D_{z'_0} H(z, x) \leq N_2, \\ |D_{(z, x)} H(z, x)| \leq N_2. \quad (5.7)$$

Indeed, the first two sets of inequalities are given by assumption and the Lipschitz continuity of $H(z, x)$ in z uniform with respect to x . If $\rho(x_0) \leq \rho_0/2$ (ρ_0 is taken from Assumption 5.3 (v)), then $B_h(x_0) \subset \partial_{\rho_0} \Omega$ and $D_{(z, x)} H(z, x)$ (on D_H) coincides with the gradient (in \mathbb{S}) of the right-hand side of (5.3), which is bounded due to Assumption 5.3 (i), implying the last inequality in (5.7). However, if $\rho_\Omega(x_0) \geq \rho_0/2$, then (for small h) $\rho_\Omega(x_0) - 3h \geq (1/2)\rho_\Omega(x_0) \geq (1/4)\rho_0$ and the last inequality in (5.7) follows from the local Lipschitz continuity of $H(z, x)$ with respect to (z, x) , which in turn is due to the uniform Lipschitz continuity with respect to z and (5.2).

The same claim holds true, obviously, for $P(z'') - K$ and for $H(z, x) \wedge (P(z'') - K)$ as well.

Next, it is convenient to continue v_h outside Ω by setting it equal to g and for a fixed $l \in \mathbb{R}^d$, such that $|l| \leq h$, introduce the function $w_h(x) = v_h(x+l)$. Observe that, owing to Lemma 4.6, both points

$$(v_h(x_0), \delta_{h,l_{\pm 1}} v_h(x_0), \dots, \delta_{h,l_{\pm m}} v_h(x_0), \Delta_{h,l_{\pm 1}} v_h(x_0), \dots, \Delta_{h,l_{\pm m}} v_h(x_0), x_0)$$

and

$$(w_h(x_0), \delta_{h,l_{\pm 1}} w_h(x_0), \dots, \delta_{h,l_{\pm m}} w_h(x_0), \Delta_{h,l_{\pm 1}} w_h(x_0), \dots, \Delta_{h,l_{\pm m}} w_h(x_0), x_0+l)$$

are in D_H . Furthermore, $x_0 \in \Omega^{2h}$ and

$$H(w_h(x_0), \delta_h w_h(x_0), \delta_h^2 w_h(x_0), x_0 + l) \vee (P_h[w_h](x_0) - K) = 0.$$

Also

$$H(v_h(x_0), \delta_h v_h(x_0), \delta_h^2 v_h(x_0), x_0) \vee (P_h[v_h](x_0) - K) = 0.$$

We subtract these two equations and, by using the arbitrariness of x_0 and what was said above in the proof, conclude that the function $u_h = w_h - v_h$ satisfies in Ω^{2h}

$$a_k \Delta_{h,l_k} u_h + b_k \delta_{h,l_k} u_h - c u_h + f = 0$$

(cf. Section 9), where $\delta/2 \leq a_k \leq 2\delta^{-1}$, $|b_k|, c \leq N_2$, $c \geq 0$, $|f| \leq N_2 h$.

Next, in Lemma 2.2 reduce δ if needed, in such a way that the new one, say δ_1 , will satisfy $N_2 \leq \delta_1^{-1}$ and $\delta_1 \leq \delta/2$, and then take Φ_0 from that lemma. Then by the comparison principle we have in \mathbb{R}^d that

$$|w_h - v_h| \leq N_2 h \sup_{\Omega} \Phi_0 + \sup_{\mathbb{R}^d \setminus \Omega^{2h}} |w_h - v_h|.$$

The last supremum is easily estimated by using (4.8) and the fact that

$$|w_h(x) - v_h(x)| \leq |v_h(x+l) - g(x+l)| + |g(x+l) - g(x)| + |g(x) - v_h(x)|.$$

This yields (5.6) for $x, y \in \mathbb{R}^d$, such that $|x - y| \leq h$.

It only remains to observe that, if $|x - y| \geq h$, one can split the straight segment between x and y into adjacent pieces of length h combined with a remaining one of length less than h and then apply the above result to each piece (recall that $v_h = g$ outside Ω). The lemma is proved.

After that our theorem is proved in exactly the same way as Theorem 8.7 of [9] on the basis of Lemma 4.6 and the fact that the derivatives of v are weak limits of finite differences of v_h as $h \downarrow 0$ (see the proof of Theorem 8.7 of [9]). One also uses the fact that there are sufficiently many pure second order derivatives in the directions of the l_i 's to conclude from their boundedness that the Hessian of v is bounded.

The following theorem will be used when, after rewriting in terms of pure second-order derivatives general fully nonlinear elliptic operators, even depending in the Lipschitz continuous way on the unknown function and its derivatives, we obtain an operator that is only locally Lipschitz as a function of the unknown function and its first-order derivatives.

Theorem 5.3. *In Assumption 5.3 (i) replace the stipulation that $H(z, x)$ “is Lipschitz continuous with respect to $z \in \mathbb{R}^{4m+1}$ with Lipschitz constant N' for any $x \in \mathbb{R}^d$ ” with $H(z, x)$ “is Lipschitz continuous with respect to z in any ball lying in \mathbb{R}^{4m+1} with constant independent of x ” and suppose that so modified Assumption 5.3 along with Assumptions 5.1 and 5.2 are satisfied. Then equation (1.10) with boundary condition $v = g$ on $\partial\Omega$ has a unique solution $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$. Furthermore, estimates (5.5) hold with N depending only on Ω , K_0 , Λ , and δ .*

Proof. Take N , that exits by Theorem 5.1, and denote by N^0 the right-hand side (5.5). Then take a smooth odd increasing function $\chi(t)$, $t \in \mathbb{R}$, such that $\chi' \leq 1$, $\chi(t) = t$ for $|t| \leq N^0$ and $\chi(t) = 2N^0 \text{sign } t$ for $|t| \geq 3N^0$ and, for $\rho \in (0, \rho_0]$, introduce

$$H^\rho(z, x) = \delta \sum_{|k|=1}^m [z_k'' - \rho^{-1} \chi(\rho z_k'')] + H(\chi(z'), \rho^{-1} \chi(\rho z''), x)$$

where

$$\begin{aligned} \chi(z') &= (\chi(z'_0), \chi(z'_{\pm 1}), \dots, \chi(z'_{\pm m})), \\ \chi(\rho z'') &= (\chi(\rho z''_{\pm 1}), \dots, \chi(\rho z''_{\pm m})). \end{aligned}$$

Obviously, H^ρ is Lipschitz continuous on $\mathbb{R}^{4m+1} \times \mathbb{R}^d$. Also, due to $0 \leq \chi' \leq 1$, it is easy to check that Assumption 5.3 (ii) is satisfied for H^ρ with the same δ . Assumption 5.3 (iii) is obviously satisfied for H^ρ with the same K_0 and perhaps smaller \bar{H} . Assumption 5.3 (v) is satisfied for H^ρ with the same ρ_0 and

$$\delta \sum_{|k|=1}^m [z_k'' - \rho^{-1} \chi(\rho z_k'')] + H(\rho^{-1} \chi(\rho z_k''))$$

in place of $H(z'')$. Finally, Assumption 5.3 (iv) is satisfied for H^ρ with the same N' since $|\rho^{-1} \chi(\rho z_k'')| \leq |z_k''|$.

By Theorem 5.1, there is a solution v^ρ of the equation

$$H^\rho[v] \vee (P[v] - K) = 0$$

in Ω with boundary data $v = g$ on $\partial\Omega$ enjoying the properties listed in Theorem 5.1. By the choice of N^0 and (5.5) we have $H^\rho[v^\rho] = H[v^\rho]$ in Ω^ρ so that v^ρ also satisfies (1.10) in Ω^ρ . Now we find a sequence $\rho_n \downarrow 0$ such that v^{ρ_n} converge uniformly in $\bar{\Omega}$ to a function v . Obviously, $v = g$ on $\partial\Omega$, $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$ and estimates (5.5) hold. By fixing m and concentrating on Ω^{ρ_m} by Theorems 3.5.15 and 3.5.9 of [8] we get that v satisfies (1.10) in Ω^{ρ_m} for any m and hence in Ω . This takes care of the existence and the estimates.

To prove uniqueness it suffices to apply the argument in Remark 1.3 to Ω^ε for small $\varepsilon > 0$ and conclude that in Ω

$$|u - v| \leq \max_{\partial\Omega^\varepsilon} |u - v|,$$

which, after sending $\varepsilon \downarrow 0$ and taking into account that u, v are continuous in $\bar{\Omega}$ and $u = v = g$ on $\partial\Omega$, yields $u = v$ in $\bar{\Omega}$. The theorem is proved.

6. GENERAL ELLIPTIC EQUATIONS WITH LIPSCHITZ CONTINUOUS H

In this section we consider equations not necessarily written in term of pure derivatives. Fix some constants $K_1, K, N' \in [0, \infty)$. Suppose that we are given a function $H(u, x)$,

$$u = (u', u''), \quad u' = (u'_0, u'_1, \dots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathbb{S}, \quad x \in \mathbb{R}^d.$$

Assumption 6.1. (i) The function $H(u, x)$ is Lipschitz continuous with respect to u'' and at all points of differentiability of H with respect to u'' we have $D_{u''}H \in \mathbb{S}_{\delta/2}$.

(ii) The number

$$\bar{H} := \sup_{u', x} (|H(u', 0, x)| - K_0|u'|) \quad (\geq 0) \quad (6.1)$$

is finite.

(iii) The function $H(u, x)$ is Lipschitz continuous with respect to u' and at all point of differentiability of $H(u, x)$ with respect to u' we have

$$|D_{u'}H(u, x)| \leq N', \quad D_{u'_0}H(u, x) \leq 0. \quad (6.2)$$

(iv) The function $H(u, x)$ is locally Lipschitz continuous with respect to x and at all point of differentiability of $H(u, x)$ with respect to x we have

$$|D_xH(u, x)| \leq N'(1 + |u|). \quad (6.3)$$

(v) For every u', x there exists a subset of \mathbb{S} of full measure at every point of which $H(u', u'', x)$ is differentiable with respect to u'' and

$$|H(u, x) - u''_{ij} D_{u''_{ij}}H(u, x)| \leq K_1(\bar{H} + K + |u'|). \quad (6.4)$$

(vi) There is a constant $\rho_0 > 0$ and a function $H(u)$ such that

$$H(u, x) = H(u) \quad (6.5)$$

for all u if $\rho(x) \leq \rho_0$.

Remark 6.1. The condition $D_{u''}H \in \mathbb{S}_{\delta/2}$ may look strange. Why do not use $D_{u''}H \in \mathbb{S}_\delta$, which is indeed imposed in Theorem 1.4? The point is that we are going to use the same $P(u'')$ in Theorems 6.1 and 1.4, but in the proof of the latter we will replace H , satisfying $D_{u''}H \in \mathbb{S}_\delta$ with the one satisfying $D_{u''}H \in \mathbb{S}_{\delta/2}$.

Here is the result we are after.

Theorem 6.1. *Under the above assumptions equation (1.10) with boundary condition $v = g$ on $\partial\Omega$ has a unique solution $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$. Furthermore, estimates (5.5) hold with N depending only on Ω , K_0 , K_1 , Λ , and δ (in particular, N is independent of N' and ρ_0).*

Remark 6.2. If we already know that equation (1.10) in Ω with boundary condition $v = g$ has a solution $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$, then the estimate (5.4):

$$\sup_{\bar{\Omega}} |v| \leq N(\bar{H} + \|g\|_{C(\Omega)}), \quad (6.6)$$

where N depends only on Λ , K_0 , δ , and $\text{diam}(\Omega)$, is obtained on the basis of the maximum principle, by repeating the proofs of Lemmas 4.2 and 4.1 with obvious changes.

Remark 6.3. The function $H(u, x)$ is locally Lipschitz continuous with respect to (u, x) , since, due to Assumption 6.1, for any $x, y \in \mathbb{R}^d$, $u, v \in \mathbb{R}^{d+1} \times \mathbb{S}$,

$$\begin{aligned} |H(u, x) - H(v, y)| &\leq |H(u, x) - H(v, x)| + N'(1 + |v|)|x - y| \\ &\leq |H(u, x) - H(u', v'', x)| + N'|u' - v'| + N'(1 + |v|)|x - y| \\ &\leq N(\delta, d)|u'' - v''| + N'|u' - v'| + N'(1 + |v|)|x - y|. \end{aligned}$$

We first prove a version of Theorem 6.1.

Lemma 6.2. *In Assumption 6.1 (v) replace (6.4) with*

$$|H(u, x) - u''_{ij} D_{u''_{ij}} H(u, x)| \leq K_1(\bar{H} + K + N_0 + [u']), \quad (6.7)$$

where

$$[u'] = |(u'_1, \dots, u'_d)|$$

and N_0 is the right-hand side of (6.6). Then the assertion of Theorem 6.1 holds true.

Proof. Without loss of generality we may assume that $K_1 > 0$ (cf. (6.4)) and define

$$I = [-K_1, K_1], \quad J = [-2K_1, 2K_1], \quad C'' = I \times \mathbb{S}_{\delta/2}, \quad B'' = J \times \mathbb{S}_{\delta/4},$$

and also recall that $H_{u''} \in \mathbb{S}_{\delta/2}$. Then for $u' \in \mathbb{R}^{d+1}$ and $y'' \in \mathbb{S}$ introduce

$$B(u', y'', x) = \{(f, l'') \in B'' : (\bar{H} + K + N_0 + [u'])f + l_{ij}y''_{ij} \leq H(u', y'', x)\}.$$

Next, recall (1.7) and for $u' \in \mathbb{R}^{d+1}$, $x \in \mathbb{R}^d$, and

$$z'' = (z''_{\pm 1}, \dots, z''_{\pm m}) \in \mathbb{R}^{2m}$$

(m is the same as in (1.7)) define

$$\begin{aligned} \mathcal{H}(u', z'', x) &= \inf_{y'' \in \mathbb{S}} \max_{(f, l'') \in B(u', y'', x)} [(\bar{H} + K + N_0 + [u'])f \\ &\quad + \sum_{|k|=1}^m \lambda_k(l'') z''_k]. \end{aligned}$$

Then by repeating word for word the beginning of Section 4 of [12], we convince ourselves that the function \mathcal{H} is measurable, Lipschitz continuous with respect to z'' with constant independent of (u', x) ,

$$H(u, x) = \mathcal{H}(u', \langle u'' l_{\pm 1}, l_{\pm 1} \rangle, \dots, \langle u'' l_{\pm m}, l_{\pm m} \rangle, x) \quad (6.8)$$

for all values of the arguments, where l_k are taken from (1.6), and at all points of differentiability of \mathcal{H} with respect to z'' we have

$$D_{z''}\mathcal{H}(u', z'', x) \in [\hat{\delta}, \hat{\delta}^{-1}]^{2m}, \quad (6.9)$$

$$(\bar{H} + K + N_0 + [u'])^{-1}[\mathcal{H}(u', z'', x) - \langle z'', D_{z''}\mathcal{H}(u', z'', x) \rangle] \in J \quad (6.10)$$

($\hat{\delta}$ is introduced in Theorem 1.3). Furthermore, \mathcal{H} is locally Lipschitz continuous with respect to (x, u') and at all points of its differentiability with respect to (x, u') we have

$$|\mathcal{H}_x(u', z'', x)| \leq N(1 + |z''| + |u'|), \quad (6.11)$$

$$|\mathcal{H}_{u'}(u', z'', x)| \leq N(1 + |u'| + |z''|), \quad (6.12)$$

where N is a constant independent of u', z'', x (for getting (6.12) assuming the first inequality in (6.2) is crucial).

Finally (what is not coming from Section 4 of [12]), the function \mathcal{H} is a decreasing function of u'_0 . This follows from the fact that, since $[u']$ is independent of u'_0 and $H(u', y'', x)$ is a decreasing function of u'_0 , for smaller values of u'_0 the set $B(u', y'', x)$ is smaller.

Now we want to apply Theorem 5.3. Since by assumption $e_i \in \Lambda$, $i = 1, \dots, d$, there is an identification of some e_i with some l_k . Let $e_i = l_{k(i)}$, for $i = 1, \dots, d$, and then for

$$z = (z', z''), \quad z' = (z'_0, z'_{\pm 1}, \dots, z'_{\pm m}) \in \mathbb{R}^{2m+1}, \quad z'' = (z''_{\pm 1}, \dots, z''_{\pm m}) \in \mathbb{R}^{2m}$$

and $x \in \mathbb{R}^d$, set

$$\hat{\mathcal{H}}(z, x) = \mathcal{H}(z'_0, z'_{k(1)}, \dots, z'_{k(d)}, z'', x).$$

Observe that, if $\rho_\Omega(x) \leq \rho_0$, $\hat{H}(z, x)$ depends only on z , since then $H(u, x)$ depends only on u by Assumption 6.1 (vi). Assumption 5.3 (iv) is satisfied for $\hat{H}(z, x)$ due to (6.12). If we take $z'' = 0$ in (6.10), then we see that

$$|\hat{H}(z', 0, x)| \leq 2K_1(\bar{H} + K + N_0 + |z'|).$$

Hence, Assumption 5.3 (iii) is satisfied for $\hat{H}(z, x)$ with

$$2K_1(\bar{H} + K + N_0) \quad \text{and} \quad 2K_1$$

in place of \bar{H} and K_0 . Assumption 5.3 (ii) is satisfied for $\hat{H}(z, x)$ with $\hat{\delta}$ in place of δ owing to (6.9). Finally, the modification of Assumption 5.3 (i) stated in Theorem 5.3 is satisfied for $\hat{H}(z, x)$ in light of (6.9), (6.11), and (6.12). Hence, all the assumptions of Theorem 5.3 are satisfied. We draw the reader's attention to the fact that the assumptions of Theorem 5.3 are satisfied with $\hat{\delta}$ in place of δ (among a few other substitutions) and that $P[u]$ in Theorem 5.3 is taken from (4.1), in accordance with which $P[u]$ in Theorems 1.4 and 6.1 is of type (4.1) with $\hat{\delta}$ in place of δ .

By Theorem 5.3 the equation

$$\hat{H}(v(x), D_{l_{k(i)}}v(x), D_{l_k}^2v(x), x) \vee (P[v](x) - K) = 0 \quad (6.13)$$

in Ω with boundary condition $v = g$ on $\partial\Omega$ has a unique solution $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$ and the estimates (5.5) hold true if the right-hand side is replaced with

$$N[K_1(\bar{H} + K + N_0) + K + \|g\|_{C^{1,1}(\mathbb{R}^d)}], \quad (6.14)$$

where N is a constant depending only on Ω , K_1 , Λ , and $\hat{\delta}$. We recall what N_0 is and estimate (6.14) from above by

$$N(\bar{H} + K + \|g\|_{C^{1,1}(\mathbb{R}^d)})$$

with N as in the statement of the theorem. Then it only remains to observe that, in light of (6.8), equation (6.13) coincides with (1.10). This proves the lemma.

Proof of Theorem 6.1. Introduce N_0 as the right-hand side of (6.6) and cut-off the range of the variable u'_0 by setting

$$H^0(u, x) = H(-N_0 \vee u'_0 \wedge N_0, u'_1, \dots, u'_d, u'', x).$$

Then, owing to (6.4),

$$|H^0(u, x) - u''_{ij} D_{u''_{ij}} H^0(u, x)| \leq K_1(\bar{H} + K + N_0 + [u']).$$

By Lemma 6.2 equation

$$H^0[v] \vee (P[v] - K) = 0 \quad (6.15)$$

in Ω with boundary condition $v = g$ on $\partial\Omega$ has a unique solution as stated in Theorem 6.1. Furthermore, since

$$|H^0(u', 0, x)| \leq \bar{H} + K_0|u'|$$

and H^0 is a decreasing function of u'_0 , by Remark 6.2 we have $|v| \leq N_0$. But then (6.15) coincides with (1.10) and the theorem is proved.

7. PROOF OF THEOREM 1.4

Remark 7.1. If we already know that there is a $v \in C(\bar{\Omega}) \cap C_{\text{loc}}^{1,1}(\Omega)$ satisfying equation (1.10) and such that $v = g$ on $\partial\Omega$, then estimate (1.11) with N depending only on d, δ, K_0 and $\text{diam}(\Omega)$ is obtained by an easy adaptation of the proofs of Lemmas 4.2 and 4.1.

Remark 7.2. Assertion (ii) follows from (i) and Theorem 1.2 of [5]. To show this, observe that (cf. the proof of Lemma 2.5)

$$H(u, x) = [H(u, x) - H(u', 0, x)] + H(u', 0, x) = a^{ij} u''_{ij} + \sum_{i=0}^d b^i u'_i + \theta \bar{H},$$

where \mathbb{S}_δ -valued $a = (a^{ij})$, \mathbb{R}^{d+1} -valued $b = (b^i)$, and $[-1, 1]$ -valued θ are certain functions of (u, x) such that $|b| \leq K_0$. It follows from the construction of P that for a constant $\kappa = \kappa(\delta, d) > 0$ we have

$$H(u, x) \leq P(u) - \kappa \sum_{i,j} |u''_{ij}| + \bar{H} + K_0|u'|.$$

Hence, if we have a solution v like in assertion (i), then

$$H[v] \leq P[v] - \kappa \sum_{i,j} |D_{ij}v| + N_0, \quad (7.1)$$

where N_0 is \bar{H} plus K_0 times the right-hand side of (1.12).

Next,

$$\max(H[v], P[v] - K) = P[v] + Q[v],$$

where $Q[v] = (H[v] - P[v] + K)_+ - K$ and, owing to (7.1), $Q[v] = -K$ if

$$-\kappa \sum_{i,j} |D_{ij}v| + N_0 \leq -K, \quad \kappa \sum_{i,j} |D_{ij}v| \geq N_0 + K.$$

If the opposite inequality holds, then

$$|Q[v]| \leq |H[v] - H(v, Dv, 0, x)| + |P[v]| + N_0 + 2K \leq NN_0, \quad (7.2)$$

where N depends only on δ and d . It follows that the inequality between the extreme terms in (7.2) holds and v is a $W_{p,\text{loc}}^2(\Omega) \cap C(\bar{\Omega})$ -solution of $P[u] = -f$, where $f = -Q[v]$ is in $L_p(\Omega)$ (actually, bounded). By Theorem 1.2 of [5] this equation with boundary data g has a solution $u \in W_p^2(\Omega)$ and

$$\|u\|_{W_p^2(\Omega)} \leq N\|f\|_{L_p(\Omega)}.$$

By uniqueness, $v = u$, and we get (1.13).

Because of Remark 7.2 below we are only dealing with assertion (i) of Theorem 1.4. The proof of it is based on Theorem 6.1 and will be achieved in several steps in the first two of which we drop Assumptions 6.1 (v), (vi), one by one.

Lemma 7.1. *In addition to the the assumptions of Theorem 1.4 let the assumptions of Theorem 6.1, apart from Assumption 6.1 (vi), be satisfied. Then the assertions of Theorem 6.1 are still true.*

Proof. Define $H(u'') = H(0, u'', 0)$ and for $\rho_0 > 0$ find a function $\zeta_{\rho_0} \in C_0^\infty(\Omega)$ such that $\zeta_{\rho_0} = 0$ on $\Omega \setminus \Omega^{\rho_0}$ and $\zeta_{\rho_0} = 1$ on $\Omega^{2\rho_0}$ and $0 \leq \zeta_{\rho_0} \leq 1$ in Ω . Then introduce

$$H^{\rho_0}(u, x) = \zeta_{\rho_0} H(u, x) + (1 - \zeta_{\rho_0}) H(u'').$$

Obviously, $H^{\rho_0}(u, x) = H(u'')$ if $\rho(x) \leq \rho_0$ and condition (6.4) is certainly satisfied for H^{ρ_0} with $2K_1$ in place of K_1 . Furthermore,

$$\begin{aligned} |H(u, x)| &\leq |H(u, x) - H(0, u'', x)| + |H(0, u'', x) - H(0, x)| \\ &\quad + |H(0, x)| \leq N'|u'| + N|u''| + \bar{H}, \end{aligned}$$

where N' is taken from (6.2) and N accounts for the Lipschitz continuity of H in u'' . It follows that condition (6.3) is satisfied for H^{ρ_0} with a different constant N' (by the way, depending on ρ_0 but this is irrelevant). Condition (6.2) is satisfied for H^{ρ_0} with the same N' and the number \bar{H}^{ρ_0} is at most twice the \bar{H} from (6.1).

By Theorem 6.1 the equation

$$\max(H^{\rho_0}[v^{\rho_0}], P[v^{\rho_0}] - K) = 0 \quad (7.3)$$

in Ω with boundary condition $v^{\rho_0} = g$ on $\partial\Omega$ has a unique solution v^{ρ_0} with the properties described in that theorem (with N from Theorem 6.1 in (5.5) multiplied by 2, because $\bar{H}^{\rho_0} \leq 2\bar{H}$). One sends $\rho_0 \downarrow 0$ and finishes the proof of existence of a solution with desired properties as in the end of the proof of Theorem 5.3. Uniqueness is also shown as there. The lemma is proved.

Lemma 7.2. *In addition to the the assumptions of Theorem 1.4 let the assumptions of Theorem 6.1, apart from Assumptions 6.1 (v), (vi), be satisfied. Then the assertions of Theorem 6.1 are still true with N in (5.5) this time independent of K_1 (which does not even enter the assumptions of the present lemma).*

Proof. Introduce

$$P_0(u'') = \max_{a \in \mathbb{S}_{\delta/2}} \operatorname{tr} au'', \quad H_K = \max(H, P_0 - K).$$

Obviously, $P_0(u'') \leq P(u'')$, so that the equation

$$\max(H_K[u], P[u] - K) = 0$$

is equivalent to (1.10). Furthermore, as is shown in Section 3 of [12] (or as follows from Section 9), the function H_K satisfies Assumption 6.1 (i) (here Remark 6.1 is relevant).

It is also shown that the function H_K satisfies Assumptions 6.1 (iii), (iv) with the same constant N' and the number $\bar{H}_K \leq \bar{H} < \infty$, so that Assumption 6.1 (ii) is also satisfied for H_K .

By Lemma 3.2 of [12], Assumption 6.1 (v) is satisfied for H_K with

$$K_1 = N(K_0 + 1),$$

where N depends only on d and δ .

Now to finish the proof of the lemma it only remains to refer to Lemma 7.1. The lemma is proved.

Remark 7.3. One might think that we could take

$$H_K(u, x) = \max(H(u, x), P(u'') - K)$$

replacing P_0 with P . However, then H_K would satisfy Assumption 6.1 (i) with $\hat{\delta}/2$ in place of $\delta/2$ and this would make Lemma 7.1 inapplicable.

We finish the proof of Theorem 1.4 by a result, that is equivalent to this theorem.

Lemma 7.3. *In addition to the the assumptions of Theorem 1.4 let the assumptions of Theorem 6.1, apart from Assumptions 6.1 (iii), (iv), (v), (vi), be satisfied. Then the assertions of Theorem 6.1 are still true with N in (5.5) independent of K_1 (which is nowhere to be found in the assumptions of the present lemma).*

The proof of this lemma is achieved by repeating the proof of Lemma 3.1 of [12] by mollifying H with respect to x and u' and using elliptic versions of the theorems about passage to the limit inside nonlinear operators (see, for instance, Section 3.5 in [8]) instead of parabolic ones, that were used in [12]. These mollifications allow one to rely on Lemma 7.2. We say more along these lines in Section 8.

8. PROOF OF THEOREMS 1.1 AND 1.2

Proof of Theorem 1.1. First assume that $g = 0$. We are going to use the following result which is the correct version of Theorem 5.4 of [11]. We would not need the correction if in [11] and here we assumed that $F_{u''} \in \mathbb{S}_\delta$. Without this assumption the claim made in the end of the proof of Theorem 5.3 of [11], on which Theorem 5.4 of [11] is based, that $F[u] = F[u] - F[0] = Lu$ for an $L \in \mathbb{L}_{\delta, K_0}$ is unsubstantiated. However, for u satisfying (1.1) we have

$$-H(u, Du, 0, x) = H[u] - H(u, Du, 0, x) = a_{ij} D_{ij} u,$$

where (a_{ij}) is an \mathbb{S}_δ -valued function and $|H(u, Du, 0, x)| \leq \bar{G} + K_0(|u| + |Du|)$. This is enough to get the correct version of Theorem 5.3 of [11] which along with interpolation theorems lead to the correct version of Theorem 5.4 of [11], which we present below, right away. Recall that $p > d$ and $\mathring{W}_p^2(\Omega)$ is the subset of $W_p^2(\Omega)$ of functions vanishing on $\partial\Omega$.

Theorem 8.1. *Let $p \in (d, \infty)$. Then there exists a constant $\theta > 0$ depending only on Ω , p , d , K_F , and δ such that if Assumption 1.3 is satisfied with this θ , then for any function $u \in \mathring{W}_p^2(\Omega)$ satisfying (1.1) we have*

$$\|u\|_{W_p^2(\Omega)} \leq N\|F[u]\|_{L_p(\Omega)} + N\|\bar{G}\|_{L_p(\Omega)} + N\|u\|_{L_p(\Omega)} + Nt_0, \quad (8.1)$$

where N depends only on Ω , R_0 , d , p , K_F , and δ .

Since $|F[u]| = |G[u]| \leq \hat{\theta}|D^2u| + \bar{G} + K_0[|u| + |Du|]$, using interpolation theorems allowing to estimate the L_p -norm of Du through the L_p -norms of D^2u and u , we immediately get (1.4) after choosing $\hat{\theta}$ sufficiently small.

In the general case introduce $\hat{g}(x) = (g(x), Dg(x), D^2g(x))$ and

$$\hat{H}(v, x) = H(v + \hat{g}(x), x), \quad w(x) = u(x) - g(x).$$

Observe that $\hat{H}[w] = 0$ in Ω (a.e.) and $w \in \mathring{W}_p^2(\Omega)$. Furthermore, for

$$\hat{G}(v, x) := \hat{H}(v, x) - F(v'', x)$$

we have

$$\begin{aligned} |\hat{G}(v, x)| &= |F(v'' + D^2g(x), x) - F(v'', x) + G(v + \hat{g}(x), x)| \\ &\leq \hat{\theta}|v''| + N|D^2g(x)| + \bar{G} + K_0(|v'_0| + |g(x)|)^2 \\ &\quad + \sum_{i=1}^d |v'_i + D_i g(x)|^2 \leq \hat{\theta}|v''| + \hat{\bar{G}} + K_0|v'|, \end{aligned}$$

where

$$\hat{\bar{G}} = N|D^2g| + \bar{G} + K_0(g|^2 + |Dg|^2)^{1/2}$$

and N depends only on K_F and d .

It follows that the above result is applicable to w which leads to (1.4) in the general case. The theorem is proved.

Proof of Theorem 1.2. First assume that $g \in C^{1,1}(\mathbb{R}^d)$ and $\omega(t) = N_0 t$ in Assumption 1.5 (ii), where N_0 is a constant. In that case we closely follow a few steps in the proof of Theorem 2.1 of [11] given in Section 6 there. Introduce a function of one variable by setting $\xi_K(t) = 0$ for $|t| \leq K$ and $\xi_K(t) = t$ otherwise, set $H_K^0(x) = \xi_K(H(0, x))$ and define

$$H_K(u, x) = H(u, x) - H_K^0(x).$$

Notice that since $F(0, x) = 0$, we have $|H_K^0| = |\xi_K(G(0, x))| \leq \xi_K(\bar{G}) \leq \bar{G}$. Also

$$H_K(0, x) = G(0, x)I_{|G(0, x)| \leq K}$$

so that $|H_K(0, x)| \leq K \wedge \bar{G}(x)$ and

$$|H_K(u', 0, x)| \leq |H_K(u', 0, x) - H_K(0, x)| + |H_K(0, x)| \leq N_0|u'| + K.$$

This shows that, for H_K , Assumption 1.7 (ii) is satisfied with N_0 in place of K_0 and an $\bar{H} \leq K$. Assumption 1.7 (i) is satisfied with the same δ and ω . By Theorem 1.4, with $P(u'')$ from (1.9), there is a solution $v_K \in W_p^2(\Omega)$ of the equation

$$\max(H_K[v_K], P[v_K] - K) = 0 \quad (8.2)$$

in Ω with boundary data g .

We want to apply Theorem 1.1 to (8.2). To this end introduce

$$\hat{H}_K(u, x) = \max(H_K(u, x), P(u'') - K),$$

$$F_K(u'', x) = \max(F(u'', x), P(u'') - K), \quad G_K(u, x) = \hat{H}_K(u, x) - F_K(u'', x).$$

Below in this section by N we denote various constants which depend only on Ω, R_0, d, p, K_0 , and δ .

Observe that

$$\begin{aligned} |G_K(u, x)| &\leq |H(u, x) - H_K^0(x) - F(u'', x)| \\ &= |G(u, x) - H_K^0(x)| \leq \hat{\theta}|u''| + K_0|u'| + 2\bar{G}(x). \end{aligned} \quad (8.3)$$

Furthermore, F_K obviously satisfies Assumption 1.3 (i) perhaps with a constant N (independent of K) in place of K_F . To check that the remaining conditions in Assumption 1.3 are satisfied take $z \in \Omega$, $r \in (0, R_0]$, the function $\bar{F} = \bar{F}_{z,r}$ from Assumption 1.3 and set

$$\bar{F}_K(u'') = \max(\bar{F}(u''), P(u'') - K).$$

Notice that

$$t^{-1}|F_K(tu'', x) - \bar{F}_K(tu'')| \leq t^{-1}|F(tu'', x) - \bar{F}(tu'')|,$$

which implies that Assumption 1.3 is satisfied indeed with the same θ, R_0 , and t_0 and, perhaps, modified $\delta > 0$ independent of K .

It follows by Theorem 1.1

$$\|v_K\|_{W_p^2(\Omega)} \leq N\|\bar{G}\|_{L_p(\Omega)} + N\|g\|_{W_p^2(\Omega)} + N\|v_K\|_{L_p(\Omega)} + Nt_0. \quad (8.4)$$

Since equation (8.2) can be rewritten as

$$a^{ij}D_{ij}v_K + b^iD_iv_K - cv_K + 2\theta\bar{G} = 0$$

(cf. Lemma 2.5), where (a^{ij}) is a certain $\mathbb{S}_{\bar{\theta}}$ -valued function, $|(b^i)| \leq K_0$, $c \geq 0$, by the Aleksandrov estimate we can eliminate $\|v_K\|_{L_p(\Omega)}$ on the right in (8.4) and conclude that

$$\|v_K\|_{W_p^2(\Omega)} \leq N\|\bar{G}\|_{L_p(\Omega)} + N\|g\|_{W_p^2(\Omega)} + Nt_0. \quad (8.5)$$

In this way we completed a crucial step consisting of obtaining a uniform control of the $W_p^2(\Omega)$ -norms of v_K .

We now let $K \rightarrow \infty$. As is well known, there is a sequence $K_n \rightarrow \infty$ as $n \rightarrow \infty$ and $v \in W_p^2(\Omega)$ such that $v_{K_n} \rightarrow v$ weakly in $W_p^2(\Omega)$. Of course, estimate (8.5) holds with v in place of v_K .

By the compactness of embedding of $W_p^2(\Omega)$ into $C(\bar{\Omega})$ we have that $v_{K_n} \rightarrow v$ also uniformly.

Next, the operator $H[u]$ fits in the scheme of Section 5.6 of [8] as long as $\omega(t) = N_0t$. In addition, by recalling that $|H_K^0| \leq \xi_K(\bar{G})$ we get

$$\begin{aligned} |H[v_K]| &= |\max(H[v_K] - H_K^0, P[v_K] - K) - H[v_K]| \\ &= |\max(0, P[v_K] - H[v_K] + H_K^0 - K) - H_K^0| \\ &\leq (P[v_K] - H[v_K] + H_K^0 - K)_+ + |H_K^0| \\ &\leq (P[v_K] - H[v_K] + H_K^0 - K)_+ + \xi_K(\bar{G}) \\ &\leq (N|D^2v_K| + N|Dv_K| + N|v_K| + \bar{G} - K)_+ + \xi_K(\bar{G}), \end{aligned}$$

so that

$$\|H[v_K]\|_{L^d(\Omega)}^d \leq NK^{d-p} \int_{\Omega} (|D^2v_K| + |Dv_K| + |v_K| + \bar{G})^p dx \rightarrow 0$$

as $K \rightarrow \infty$. By combining all these facts and applying Theorems 3.5.15 and 3.5.6 of [8] we conclude that $H[v] = 0$ and this finishes the proof of the theorem if $\omega = N_0t$ and $g \in C^{1,1}(\mathbb{R}^d)$. If $g = 0$, in this way we obtain a slightly different proof of the existence part in Theorem 2.1 of [11].

To pass to the general ω , take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$, which integrates to one and has support in the unit ball centered at the origin and define $H^n(u, x)$ as the convolution of $H(u, x)$ and $n^{d+1}\zeta(nu')$ performed with respect to u' . Keep $F_n = F$. As is easy to see, for each n , H_n satisfies Assumption 1.2 with the same K_0 and $\hat{\theta}$ and $\bar{G} + 1/n$ in place of \bar{G} . Furthermore, for any $k = 0, \dots, d$

$$\begin{aligned} H_{u'_k}^n(u', u'', x) &= n \int_{\mathbb{R}^{d+1}} H(u' - v'/n, u'', x) \zeta_{u'_k}(v') dv' \\ &= n \int_{\mathbb{R}^{d+1}} [H(u' - v'/n, u'', x) - H(u', u'', x)] \zeta_{u'_k}(v') dv'. \end{aligned}$$

It follows that

$$|H_{u'_k}^n(u, t, x)| \leq n\omega(1/n)|B_1|_{\mathbb{R}^{d+1}} \sup |D\zeta|,$$

where $|B_1|_{\mathbb{R}^{d+1}}$ is the volume of the unit ball in \mathbb{R}^{d+1} , so that H^n also satisfies Assumption 1.5 (ii) with a constant N , depending on n , times t in place of ω . Assumptions 1.5 (i) and 1.1 are obviously satisfied for H^n . Since we did not change F , Assumption 1.3 is satisfied and by the first part of the proof, there exists $v^n \in W_p^2(\Omega)$ such that $H^n[v^n] = 0$ in Ω and $v^n = g$ on $\partial\Omega$. Furthermore, (8.5) holds with v^n in place of v_K , $\|\bar{G}\|_{L_p(\Omega)} + 1$ in place of $\|\bar{G}\|_{L_p(\Omega)}$, and N independent of n .

Owing to embedding theorems, $W_p^2(\Omega) \subset C^{1+\gamma}(\Omega)$, where $\gamma > 0$, the sequence $\{v^n, Dv^n\}$, being uniformly bounded and uniformly continuous, has a subsequence uniformly in Ω converging to v, Dv , where $v \in W_p^2(\Omega)$. For simplicity of notation we suppose that the whole sequence v^n, Dv^n converges. Of course, $v = g$ on $\partial\Omega$.

Observe that for $m \geq n$

$$\check{H}^n[v^m] \geq 0 \quad (8.6)$$

in Ω (a.e.), where

$$\check{H}^n(u, x) := \sup_{k \geq n} H^k(v^k(x), Dv^k(x), u'', x).$$

In light of (8.6) and the fact that the norms $\|v^n\|_{W_p^2(\Omega)}$ are bounded, by Theorems 3.5.15 and 3.5.9 of [8] we have

$$\check{H}^n[v] \geq 0 \quad (8.7)$$

in Ω (a.e.).

Now we notice that

$$\begin{aligned} |H^k(u, x) - H(u, x)| &\leq \omega(1/k), \\ |H^k(v^k(x), Dv^k(x), D^2v(x), x) - H(v(x), Dv(x), D^2v(x), x))| \\ &\leq \omega(|v^k - v|(x) + |Dv^k - Dv|(x)) + \omega(1/k), \end{aligned}$$

which along with what was said above implies that

$$\hat{H}^n[v] \geq -\varepsilon_n \quad (8.8)$$

in Ω (a.e.), where the functions $\varepsilon_n \rightarrow 0$ in Ω (even uniformly) and

$$\hat{H}^n(u, x) := \sup_{k \geq n} H^k(u, x),$$

By letting $n \rightarrow \infty$ in (8.8) and using that $H^k(u, x) \rightarrow H(u, x)$ as $k \rightarrow \infty$ for any (u, x) , we conclude that $H[v] \geq 0$ in Ω (a.e.).

The inequality $H[v] \leq 0$ in Ω (a.e.) is obtained by similar arguments starting with

$$\inf_{k \geq n} H^k(v^k(x), Dv^k(x), u'', x).$$

The passage from $g \in C^{1,1}(\mathbb{R}^d)$ to $g \in W_p^2(\Omega)$ is achieved by mollifying g and using a very simplified version of the above arguments. The theorem is proved.

9. APPENDIX

Here are two results used mostly inexplicitly in various combinations at various places in the article.

Lemma 9.1. *Let $F(u)$ be a real-valued Lipschitz continuous function defined in \mathbb{R}^n . Assume that there is a convex closed bounded set $A \subset \mathbb{R}^n$ and a set of full measure $D'_F \subset \mathbb{R}^n$ such that at all points $u \in D'_F$ the function F is differentiable and $DF(u) \in A$. Then for any $u, v \in \mathbb{R}^n$ there exists an $a \in A$ such that*

$$F(u) - F(v) = a^i(u^i - v^i). \quad (9.1)$$

In particular,

$$\min_{a \in A} [a^i(u^i - v^i)] \leq F(u) - F(v) \leq \max_{a \in A} [a^i(u^i - v^i)]. \quad (9.2)$$

Proof. Fix $v \in \mathbb{R}^n$. By using the Fubini theorem in polar coordinates we obtain that, for almost all points $\omega \in \partial B_1$, $v + t\omega \in D'_F$ for almost all t . In addition, $F(v + t\omega)$ is a Lipschitz and absolutely continuous function of t on that interval. It follows that, for almost all points $\omega \in \partial B_1$, for almost all t we have

$$\partial_t F(v + t\omega) = \omega^i [D_{u^i} F](v + t\omega)$$

and for all $t > 0$

$$F(v + t\omega) = F(v) + ta^i \omega^i,$$

where

$$a^i = \frac{1}{t} \int_0^t [D_{u^i} F](v + s\omega) ds.$$

Since A is closed and convex, $a \in A$, and we obtain (9.1) for almost all $u \in \mathbb{R}^n$. Then by compactness of A and the continuity of F we extend (9.1) to all $u, v \in \mathbb{R}^n$. The lemma is proved.

The following is in a sense a converse statement to Lemma 9.1.

Lemma 9.2. *Let A be as in Lemma 9.1 and let $F(u)$ be a real-valued function such that for all $u, v \in \mathbb{R}^n$*

$$F(u) - F(v) \leq \max_{a \in A} [a^i(u^i - v^i)] =: U(u - v). \quad (9.3)$$

Then F is Lipschitz continuous on \mathbb{R}^n and at all points $u \in \mathbb{R}^n$, at which F is differentiable, we have $DF(u) \in A$.

Proof. Obviously $|U(w)| \leq N|w|$, where N is a constant independent of w . By interchanging u and v in (9.3) we see that $F(u) - F(v) \geq -U(v - u)$, which along with (9.3) yields the Lipschitz continuity of F . In addition, if F

is differentiable at $v \in \mathbb{R}^n$, then $F(u) - F(v) = D_{u^i}F(v)(u^i - v^i) + o(|u - v|)$ and using (9.3) divided by $|u - v|$ and setting $u - v \rightarrow 0$ we get

$$D_{u^i}F(v)\omega^i \leq \max_{a \in A}[a^i\omega^i]$$

for any unit ω . This is only possible if $DF(v) \in A$ owing to the fact that A is closed, bounded, and convex. The lemma is proved.

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